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Analytic number theory, by Henryk Iwaniec and Emmanuel Kowalski, Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004, xii+618 pp., US$99.00, ISBN 0-8218-3633-1

The book under review is an important comprehensive account of modern analytic number theory. It is written in a relaxed, personal style with graduate students in mind, but there is plenty in it for the expert. The book is often witty, and in a long and unusually useful introduction there is a panoramic survey of the subject in which the authors set out their reasons for the choices of topics they have made. The 26 chapters give a masterly introduction to many of the most exciting areas of contemporary research. Of course, there is not room for everything, but I think it is fair to say that the authors' objectives are still the major themes of analytic number theory. They demonstrate how the huge diversity of analytic methods have made an impact on the field. They write with authority and show their mastery of all the material on every page. One might say that they do for modern theory what Landau did for it in the first half of the last century, no mean achievement given all that has been accomplished since Landau's time.

The book opens with five introductory chapters, one connected to the other and meant to be read consecutively, on the classic topics of arithmetic functions, elementary prime number theory, characters, summation formulas, and the analytic theory of L-functions. These chapters are suitable for supplementary reading for a beginning graduate course in analytic number theory. The discussion in the later chapters is invigorating and serious. It leads the reader through an abundance of highly nontrivial and sophisticated technical details—for example, on exponential sums, Kloosterman sums, equidistribution theory, averages of Fourier coefficients, and correlation of zeros of the Riemann zeta-function—that are ideal for graduate topics courses that stimulate further study in specialized monographs and research articles. One of the delightful features of this book is that inside each section are interesting exercises that provide additional information about the subject and that provoke and encourage one to make one's own discoveries in the subject at hand. The mathematical prerequisites call for some acquaintance with the fundamentals of elementary number theory, advanced calculus involving inequalities, complex analysis and integration, and abstract algebra.

Euler is credited with being the first mathematician to use analytical arguments for the purpose of investigating properties of integers, by constructing generating power series, although, at least in principle, ideas relating analysis with number theory can be traced before him. It is worth mentioning that, before Euler, a number of great mathematicians had attempted to establish exact formulas for the computation of the transcendental numbers π and e. In retrospect, their power series expansions and continued fraction expansions for these numbers show a clear and conscientious effort of using tools from analysis in order to better understand the nature of numbers. Euler pushed further the use of methods from analysis to questions about number theory. His use of the zeta-function and the corresponding product over primes not only proved the existence of infinitely many primes, which

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was known to Euclid, but it also established a quantitative relation, namely that the sum of inverses of primes is divergent. This paved the way for the appearance of asymptotic formulas in number theory, and constitutes the beginning of analytic number theory.

The impact of Dirichlet's creation of the theory of L-functions for characters and his theorem on the existence of infinitely many primes in any given arithmetic progression $\{qn + a, n \in \mathbb{N}\}$, with q and a being relatively prime positive integers, immediately led to the appearance of analytic number theory as a mathematical discipline. To attack the above problem Dirichlet made use of what are now called Dirichlet characters $\chi(\bmod q)$ and the corresponding Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{\chi(p)}{p^s}},$$

for $s > 1$. A combination of such L-functions is then used to isolate those prime numbers which belong to the given arithmetic progression,

$$\frac{1}{\varphi(q)} \sum_{\chi(\bmod q)} \overline{\chi}(a) \log L(s,\chi) = \sum_{\substack{p \text{ prime} \\ p^m \equiv a (\bmod q)}} \sum_{m \geq 1} \frac{1}{mp^{ms}}. \tag{1}$$

Relations like this make one realize how well number theory and analysis can go hand in hand. In order to exploit (1), one lets $s \to 1^+$. Then the right side of (1) is

$$\sum_{p \equiv a (\bmod q)} \frac{1}{p^s} + \mathcal{O}(1),$$

while on the left the principal character $\chi = \chi_0$ gives the main, expected contribution, and one is left with the problem of showing that $L(s,\chi) \neq 0$ for all the other characters modulo q. Dirichlet solved this problem brilliantly, thus completing his proof of the infinitude of primes in arithmetic progressions, and at the same time producing his famous class number formulas for both imaginary and real quadratic fields. In fact, the lower bound for $L(1,\chi)$ for a quadratic character χ that follows from Dirichlet's work, in the sense that it follows via Dirichlet's work from the fact that the class number is ≥ 1, is even today close to the best effective lower bound known. The striking work of Goldfeld and Gross-Zagier solved the so-called class number problem. Goldfeld [3] gave an effective lower bound for the class number $h(D)$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{2})$, the quality of which depends on the rank of an auxiliary elliptic curve and its satisfying the Birch and Swinnerton-Dyer Conjecture. Gross and Zagier ([5], [6]) produced a curve with the required properties. Oesterlé [Oe] reduced the implied constants in Goldfeld's work and obtained the clean estimate

$$h(D) > \frac{1}{55}(\log|D|) \prod_{p|D} \left(1 - \frac{[2\sqrt{p}]}{p+1}\right).$$

The authors present Goldfeld's approach, with some variations of their own, and also survey the construction of Gross and Zagier.

Returning to $L(s,\chi)$, let us mention Siegel's much stronger, although ineffective, lower bound, which states that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that if χ is a real primitive character to the modulus q, then

$$L(1,\chi) > \frac{C(\varepsilon)}{q^{\varepsilon}}.$$

The book under review contains much more material on this subject. The authors thoroughly present the sharpest results known, and analyze the strength of present-day techniques. The entire discussion on the topics surrounding the Siegel zero problem is carefully and at great length treated by Iwaniec and Kowalski and constitutes one of the highlights of the book. In this connection one should also mention the striking work of Iwaniec and Sarnak [8] on nonvanishing of central values of automorphic L-functions and Landau-Siegel zeros.

Returning to the beginnings of analytic number theory, after Dirichlet, Riemann's memoir [14] led to a huge advance of the theory. The only paper written by Riemann on the subject, it has shaped the evolution of analytic number theory ever since. The complex function $\zeta(s)$ that is now called the Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for $\mathrm{Re}(s) > 1$. In the same half plane $\zeta(s)$ has an Euler product,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}\,.$$

Riemann proved that $\zeta(s)$ can be continued analytically over the whole plane and is then meromorphic, its only pole being a simple pole at $s = 1$ with residue 1. He also proved that $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{1}{2}s}\Gamma\Big(\frac{1}{2}s\Big)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\Big(\frac{1}{2}(1-s)\Big)\zeta(1-s).$$

Next, Riemann made a number of very remarkable conjectures on an asymptotic formula for the number $N(T)$ of zeros of $\zeta(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$, with $0 < \mathrm{Im}(s) \le T$; on the product representation for the entire function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$; on the explicit formula for $\pi(x) - \mathrm{Li}(x)$ for all $x > 1$; and on the famous Riemann Hypothesis that the zeros of $\zeta(s)$ in the critical strip all lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Except for the Riemann Hypothesis, the other conjectures were proved by von Mangoldt and Hadamard and culminated in the complete proof by Hadamard and de la Vallée Poussin of the prime number theorem, which states that the number $\pi(x)$ of prime numbers less than or equal to x satisfies the asymptotic relation

$$\pi(x) \sim \frac{x}{\log x}\,,$$

as $x \to \infty$. The remaining conjecture, the Riemann Hypothesis, is currently listed among the Millennium Problems by the Clay Institute (in this connection the reader is referred to Bombieri's [1] and Sarnak's [15] presentations of the Riemann Hypothesis). The Riemann zeta function is only one entity in an entire world of L-functions. Iwaniec and Kowalski embed the discussion of the Riemann zeta function in the context of abstract L-functions.

The reader who is already acquainted with the classical theory of the Riemann zeta-function will quickly realize that the study of these more general objects also demands the invention of new methods. For example, to obtain zero-free regions for L-functions of degree > 1, one cannot use the same method as for Dirichlet L-functions, the key ingredient in the case of higher degree L-functions being the Rankin-Selberg L-function. The Riemann zeta-function $\zeta(s)$, together with the

Dirichlet L-functions $L(s,\chi)$, gives all the degree one L-functions (i.e., in the product over primes each local factor at p is the inverse of a polynomial of degree one in p^{-s}). The general L-function of degree m comes from an automorphic form on the general linear group of m by m invertible matrices, GL_m. As an example, the discriminant

$$\triangle(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24} := \sum_{n=1}^{\infty}\tau(n)e^{2\pi inz},$$

for z in the upper half plane, is a holomorphic cusp form of weight 12 for the modular group, that is,

$$\triangle\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\triangle(z)$$

for $a,b,c,d \in \mathbb{Z}, ad-bc=1$. The associated L-function is of degree two,

$$L(s,\triangle) = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^{11/2}}n^{-s} = \prod_p\left(1-\frac{\tau(p)}{p^{11/2}}p^{-s}+p^{-2s}\right)^{-1}.$$

The function $L(s,\triangle)$ is entire and satisfies the functional equation

$$\Lambda(s,\triangle) := \Gamma_{\mathbb{R}}\left(s+\frac{11}{2}\right)\Gamma_{\mathbb{R}}\left(s+\frac{13}{2}\right)L(s,\triangle) = \Lambda(1-s,\triangle),$$

where

$$\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2).$$

More generally, let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and let π be an automorphic cuspidal representation of $GL_m(\mathbb{A})$ with central character χ. The representation π is equivalent to $\otimes_v\pi_v$ with $v=\infty(\mathbb{Q}_\infty=\mathbb{R})$ or $v=p$ and π_v an irreducible unitary representation of $GL_m(\mathbb{Q}_v)$. For each prime p one forms the local factor

$$L(s,\pi_p) = \prod_{j=1}^{m}(1-\alpha_{j,\pi}(p)p^{-s})^{-1},$$

with $\alpha_{j,\pi}(p)$ determined by π_p. For $v=\infty, \pi_\infty$ determines parameters $\mu_{j,\pi}(\infty)$ such that

$$L(s,\pi_\infty) = \prod_{j=1}^{m}\Gamma_{\mathbb{R}}(s-\mu_{j,\pi}(\infty)).$$

The standard L-function associated to π is defined by

$$L(s,\pi) = \prod_p L(s,\pi_p),$$

and the completed L-function is given by

$$\Lambda(s,\pi) = L(s,\pi_\infty)L(s,\pi).$$

One shows that $\Lambda(s,\pi)$ is entire and satisfies a functional equation

$$\Lambda(s,\pi) = \epsilon_\pi N_\pi^{\frac{1}{2}-s}\Lambda(1-s,\tilde{\pi}),$$

where $N_\pi \geq 1$ is the conductor of π, ϵ_π is of modulus 1 and is computable in terms of Gauss sums, and $\tilde{\pi}$ is the contragradient $\tilde{\pi}(g) = \pi({}^tg^{-1})$. General conjectures of Langlands assert that these standard L-functions multiplicatively generate all L-functions (in particular Dedekind Zeta Functions, Artin L-functions, Hasse-Weil Zeta Functions,...). The Grand Riemann Hypothesis states that for any π as above, the zeros of $\Lambda(s,\pi)$ all lie on the line $\Re(s)=\frac{1}{2}$. Hilbert and Polya suggested that there might be a spectral interpretation of the zeros of the zeta function in which the corresponding operator is self-adjoint. The spectral nature of the zeros emerges clearly when studying the local statistical fluctuations of the high zeros of a given $\Lambda(s,\pi)$ or the low-lying zeros of a given family of L-functions. These statistical distributions are apparently dictated by random ensembles (see [10]). For a given $\Lambda(s,\pi)$ the local fluctuations of high zeros are universal and follow the laws of fluctuations of the "Montgomery-Odlyzko Law". For low-lying zeros in a family there is a symmetry type that one can associate with the family from which the densities and fluctuations can be predicted. Numerical experimentations starting with [12] have given further striking confirmation of this statistical fluctuation phenomenon. The spectral methods of automorphic forms have a huge impact on current analytic number theory. Iwaniec and Kowalski succeeded in presenting a concise foundation of the theory (for further reading one should consult the books of Iwaniec [9] and Sarnak [16]) and at the same time giving as applications a few gems, such as the estimation of sums of Kloosterman sums, which have applications to many problems, and the equidistribution of roots of quadratic congruences of prime moduli.

The two most important tools in the additive theory of numbers (a loosely defined category of topics which includes Waring's problem, the Twin Prime problem and Goldbach's conjecture) are sieve methods and the circle method. As comprehensive accounts are available in monographs [4, 7, 18], the authors present only basic introductions to these subjects. The book covers many important topics in analytic number theory, and the reader interested in the current state of knowledge in a specific subject has much to learn from Kowalski and Iwaniec's presentation. Bilinear forms and the large sieve, exponential sums (including the classical methods of Weyl, Van der Corput and Vinogradov), zero density estimates, exponential sums in finite fields, character sums, sums over primes, the circle method, primes in arithmetic progressions (including the Bombieri-Vinogradov theorem), the Goldbach problem, the least prime in an arithmetic progression, critical zeros of the Riemann zeta function, and central values of L-functions are some of the important topics in analytic number theory which are carefully treated in depth in the book. The book closes with a detailed and up-to-date bibliography. This is an impressive work and can be recommended for classroom use at the graduate level and for reference.

There are several classical and notable books that introduce new researchers to the basic results, methods, and terminology of analytic number theory and, at the same time, serve the dual purpose of textbooks for graduate students in many different fields of mathematics. Considering that the literature in the field has grown prodigiously in the last hundred years, none to date has been written so ambitiously and on such a grand scale as the book under review. I predict that the book will have a major impact on research and on aspiring young mathematicians.

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