

## KNOWN AND UNKNOWN RESULTS ON ELLIPTIC BOUNDARY PROBLEMS

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In a survey article by W. N. Everitt and L. Markus [EM05] in the October 2005 issue of *Bulletin AMS*, the authors raise some questions on boundary problems for elliptic partial differential operators. The article is primarily concerned with ODE where the choice of boundary conditions is finite dimensional, but a secondary purpose is to extend their points of view to elliptic PDE where the possible boundary conditions range in infinite-dimensional spaces, and here the authors present a number of what they consider open problems.

The purpose of this note is to show how answers to the problems, as well as much more extensive results, can be found in the existing literature.

Section 2.2 of [EM05] presents the partial differential operator to be considered, namely the Laplacian  $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$  on the unit disk  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ . With  $A$  denoting  $-\Delta$  applied in the distribution sense, the authors consider several operators acting like  $A$  (realizations of  $A$ ):

- $T_0$  is the minimal operator, with domain  $D(T_0) = \mathring{W}^2(\Omega)$ ,
- $T_1$  is the maximal operator, with domain  $D(T_1) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}$ ,
- $T_{\text{Dir}}$  is the Dirichlet realization, with domain  $D(T_{\text{Dir}}) = W^2(\Omega) \cap \mathring{W}^1(\Omega)$ .

It has been known for many years that  $T_{\text{Dir}}$  is selfadjoint in  $L_2(\Omega)$  with a positive lower bound, that  $T_0$  is closed, densely defined and symmetric with the same lower bound, and that  $T_1 = T_0^*$ .  $T_1$  has an infinite-dimensional nullspace consisting of the harmonic functions in  $L_2(\Omega)$ ,

$$(1) \quad L_2^\Delta(\Omega) = \{u \in L_2(\Omega) \mid Au = 0\}.$$

In Section 2.3, the authors introduce one more realization,  $T_{\text{Har}}$ , that they find mysterious; its domain is

$$(2) \quad D(T_{\text{Har}}) = \mathring{W}^2(\Omega) \dot{+} L_2^\Delta(\Omega),$$

and it is selfadjoint. They prove by examples, referring to [EMP05] for details, that  $D(T_{\text{Har}})$  is not contained in  $W^1(\Omega)$  and that there are elements of  $D(T_{\text{Har}})$  that do not have pointwise radial limits for  $r \nearrow 1$ . They claim on p. 480:

“It is an unsolved problem as to whether the operator  $T_{\text{Har}}$ , as a self-adjoint extension of the Laplace operator in the unit disk, is unique in some noteworthy

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way. For instance, can  $T_{\text{Har}}$  be characterized, among all self-adjoint operators for the Laplace differential expression, or even among some interesting subclass, by mathematically significant intrinsic properties?"

**Answer 1.** In the setting of abstract operators in Hilbert space,  $T_{\text{Har}}$  is the solution by J. von Neumann [N29] of the problem of finding a selfadjoint extension of a given symmetric, densely defined positive operator. It was found before the Friedrichs extension [F34], which in the present situation equals  $T_{\text{Dir}}$ . M. G. Krein showed in [K47] that the full set of selfadjoint nonnegative extensions of  $T_0$  can be characterized as the operators lying between the “hard” extension (here  $T_{\text{Dir}}$ ) and the “soft” extension (here  $T_{\text{Har}}$ ) in a sense defined via sesquilinear forms. So indeed  $T_{\text{Har}}$  is intrinsically characterized as Krein’s soft extension of  $T_0$ .

In terms of boundary conditions, Grubb [G68] showed that  $T_{\text{Har}}$  represents the nonlocal boundary condition

$$(3) \quad Mu = 0, \text{ where } M = \gamma_1 - P\gamma_0;$$

here  $\gamma_j u = \partial_n^j u|_{\partial\Omega}$ , the  $j$ 'th normal derivative, and  $P$  is the “Dirichlet-to-Neumann” operator over  $\partial\Omega$ , mapping  $\gamma_0 u$  into  $\gamma_1 u$  when  $u \in L_2^{\Delta}(\Omega)$ . These statements build on the result of Lions and Magenes from 1961-62, reproduced in the book [LM68], that the trace operators  $\gamma_j$  extend to continuous mappings from  $D(T_1)$  to  $W^{-j-\frac{1}{2}}(\partial\Omega)$ ,  $j = 0, 1$ . (Another source is to apply Hörmander [H63], Th. 4.3.1, 2.5.6, to  $L_2^{\Delta}(\Omega)$ , “partial hypoellipticity at the boundary”.) Vainberg and Grushin showed in [VG67] that  $P$  is a pseudodifferential operator of order 1. This also follows from Seeley [S66] and Hörmander [H66], and is accounted for in detail in Grubb [G71]. It is shown in [G68] that  $D(T_{\text{Har}}) \not\subseteq W^s(\Omega)$  for all  $s > 0$ .

Next, Everitt and Markus write on p. 480:

“The question of uniqueness for the operator  $T_{\text{Har}}$  also leads to another problem. If  $T_{\text{Har}}$  is unique, can all other self-adjoint extensions of the Laplace operator on the unit disk be determined by applying generalized boundary conditions to functions in the maximal domain  $D(T_1)$ , on the boundary  $\partial\Omega$ , using the properties of the Trace Theorem?” (The Trace Theorem in [EM03] defines  $\gamma_j$  on  $W^l(\Omega)$  for  $l > j$ .)

**Answer 2.** The full set of selfadjoint extensions of  $T_0$  was characterized in [G68] in terms of boundary conditions, using the extended trace theorem of [LM68] indicated above. The selfadjoint extensions  $\tilde{T}$  correspond 1–1 to the pairs  $\{X, L\}$ , where  $X$  is a closed subspace of  $W^{-\frac{1}{2}}(\partial\Omega)$  and  $L$  is a selfadjoint operator from  $X$  to the dual space  $X'$  (with  $D(L) \subset X$ ). Here, when  $\tilde{T}$  corresponds to  $\{X, L\}$ , it represents the boundary condition

$$(4) \quad \gamma_0 u \in D(L), \quad (Mu)|_X = L\gamma_0 u.$$

The characterization is worked out for general elliptic operators on general smooth sets  $\Omega$  and deals with many properties other than selfadjointness. The scale of semibounded realizations arising from sesquilinear forms, as initiated by Krein, is considered in [G70].

In Section 5 of [EM05], the authors present a number of open problems. Most of these are concerned with ODE. The problem posed in PDE, Problem 5.7, is whether  $T_{\text{Har}}$  is globally determined when  $\partial\Omega$  is connected.

**Comment 1.** Realizations determined by purely differential boundary conditions are considered to be locally determined. As recalled above in Answer 1,  $T_{\text{Har}}$  is determined by the boundary condition (3). The operator  $P$  is truly pseudodifferential; for the Laplace operator, its principal symbol is  $-|\xi'|$ , and there are similar nonpolynomial symbols in more general cases (detailed formulas can be found in [G71], Appendix). Then  $P$  is in general nonlocal, with a distribution kernel not supported in the diagonal. In this sense, the boundary condition for  $T_{\text{Har}}$  is global.

Quotations from the end of the article, p. 498:

“... the Harmonic operator remains elusively unspecified by any kind of boundary evaluations.”

**Comment 2.** No, it is determined by (3).

“Recent research (see [16]) has uncovered a continuum of distinct analogues of the operator  $T_{\text{Har}}$ , [...] each of which has a domain not entirely contained within the Sobolev space  $W^2(\Omega)$ . Hence these domains must contain some functions having no boundary values in the sense of the trace map [...].

“It remains an open problem to produce a coherent, and plausibly explicit, description of the set of all self-adjoint operators generated by the Laplace differential expression on the unit disk [...].”

**Comment 3.** No, the problem is not open; see Answer 2 above. All the functions in the maximal domain have boundary values in the sense of the extended trace map of [LM68]. Domains not contained in  $W^2(\Omega)$  are abundant, in view of the 1–1 correspondence between selfadjoint realizations  $\tilde{T}$  and arbitrary pairs  $\{X, L\}$ .

For a fuller picture, we include some further references that are relevant in this context. The grounds for the abstract part of the characterization were laid in Vishik [V52] and Birman [B56], in addition to [K47]. Some of our results were rediscovered by Alonso and Simon [AS80] (addendum [AS81]) and taken up again in connection with a spectral asymptotics question in [G83].

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Gerd Grubb is a professor at the University of Copenhagen, Denmark. She has received the Hermer’s Prize and the Marie Lønggaard Award, and is a docteur honoris causa at Université de Reims.

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