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Spectral functions in mathematics and physics, by Klaus Kirsten, Chapman & Hall/CRC, Boca Raton, FL, 2002, xiv+382 pp., US$99.95, ISBN 1-58488-259-X

Asymptotic formulae in spectral geometry, by Peter B. Gilkey, Studies in Advanced Mathematics, vol. 43, Chapman & Hall/CRC, Boca Raton, FL, 2004, viii+304 pp., US$99.95, ISBN 1-58488-358-8

The one-dimensional heat equation is a staple of mathematical methods courses for engineers and scientists. The temperature in a uniform bar of length L is governed, after rescaling of coordinates to eliminate constants specific to the material, by the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ together with some boundary conditions, such as any of these:

$$\text{Dirichlet:} \quad u(t,0) = 0 = u(t,L) \tag{1}$$

$$\text{Neumann:} \quad \frac{\partial u}{\partial x}(t,0) = 0 = \frac{\partial u}{\partial x}(t,L) \tag{2}$$

$$\text{Mixed:} \quad u(t,0) = 0 = \frac{\partial u}{\partial x}(t,L) \tag{3}$$

If the initial temperature distribution is $u(0,x) = f(x)$, then the solution can be found as

$$u(t,x) = \int_0^L K(t,x,y) f(y)\,dy$$

in terms of a function K called the *heat kernel*.

There are two complementary ways to construct the heat kernel. First, it has the *spectral expansion*

$$K(t,x,y) = \sum_{j=1}^{\infty} \phi_j(x)\phi_j(y) e^{-\lambda_j t}, \tag{4}$$

where the ϕ and λ are the normalized eigenfunctions and eigenvalues of the Laplacian operator, e.g., $\phi_j(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{j\pi x}{L}\right)$ for boundary conditions (1). The second is the *method of images*: If we were solving the equation on the whole real line, the kernel would be

$$K_0(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}.$$

It is then easy to see that

$$K(t,x,y) = \sum_{n=-\infty}^{\infty} [K_0(t,x,y-2nL) \pm K_0(t,x,-y+2nL)] \tag{5}$$

are the kernels that satisfy (2) (upper sign) and (1) (lower sign).

From (5) we can form the *trace*

$$K(t) \equiv \int_0^L K(t,x,x)\,dx = \frac{L}{\sqrt{4\pi t}} \pm \frac{1}{2} + O(t^{\infty}), \tag{6}$$

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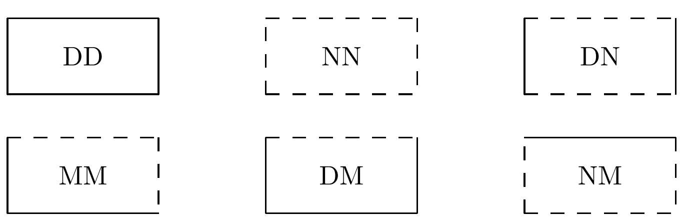
where the error term decreases exponentially as $t \to 0$. (All the image terms in (5) go to 0 pointwise, but the limit is nonuniform and the integrals of the two closest images produce the $\pm\frac{1}{2}$.) On the other hand, from (4) we have

$$K(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}. \tag{7}$$

The combination of (7) and (6) displays a connection between the *spectrum* $\{\lambda_j\}$ and the *geometry* (in a generalized sense) of the original problem, since (6) reflects both the length of the interval, L, and the nature of the boundary conditions.

Modern research mathematicians, of course, are interested in less trivial differential equations. Before going there, however, one might ask why a mathematician should be interested in heat kernels at all. The obvious answer is, "To study heat conduction, of course." Nevertheless, for decades there has been intense interest in heat kernels, especially their asymptotic expansions at small t, on the part of algebraic topologists and differential geometers who are not motivated by engineering applications, and also theoretical physicists working in the esoteric reaches of high-energy particle physics and general relativity. The core reason is already indicated in the example above: The heat kernel sits in the middle between a differential operator itself and its geometric setting, on the one hand, and the eigenvalues and other spectral apparatus associated with that operator, on the other. It is a medium through which information about the geometry is translated into information about the spectrum and vice versa. The equality of (6) and (7), generalized to other second-order operators in one dimension, implies that as $\lambda \to \infty$ the number of eigenvalues less than λ, $N(\lambda)$, must follow *Weyl's law*, $N(\lambda) \sim \frac{L}{\pi}\lambda^{1/2}$. Inversely, as popularized by M. Kac's famous "drum" question [4], knowledge of the eigenvalues puts constraints on the geometry. Heat-kernel expansions provide the now preferred route to proving the famous *index theorem*, which relates the difference in the number of eigenvectors of eigenvalue 0 of two operators A^*A and AA^* to the topology of the manifold and vector bundle where A acts [1, 2, 3].

Two techniques that have proved to be very useful for the efficient calculation of heat-kernel expansions can be demonstrated in the example of the two-dimensional heat equation, $\frac{\partial u}{\partial t} = \nabla^2 u$, in rectangles with either the Dirichlet or the Neumann condition on each edge. Apart from interchange of the two side lengths, there are only six different cases:



(Dashed lines indicate Neumann boundaries. M stands for "mixed". Let's agree that these rectangles have length L_1 and width L_2.)

The first crucial observation is that in such a problem the horizontal and vertical coordinates decouple, so that the heat kernel is simply the product of the two onedimensonal kernels. Therefore, the same is true of the trace. For example, for the rectangle with two Dirichlet sides of length L_2 and two Neumann sides of length L_1, one has (dropping the small error terms)

$$K_{DN}(t) = K_D(t)K_N(t) \equiv \left[\frac{L_1}{\sqrt{4\pi t}} - \frac{1}{2}\right]\left[\frac{L_2}{\sqrt{4\pi t}} + \frac{1}{2}\right] = \frac{L_1 L_2}{4\pi t} + \frac{L_2 - L_1}{4\sqrt{\pi t}} - \frac{1}{4}. \tag{8}$$

(The L_1 appears in K_D because what is relevant there is the *separation* between the Dirichlet boundaries.) Similarly,

$$K_{DD}(t) = \frac{L_1 L_2}{4\pi t} - \frac{L_1 + L_2}{4\sqrt{\pi t}} + \frac{1}{4}, \quad K_{NN}(t) = \frac{L_1 L_2}{4\pi t} + \frac{L_1 + L_2}{4\sqrt{\pi t}} + \frac{1}{4}. \tag{9}$$

Next, one observes that (8)–(9) all follow the pattern

$$K(t) = \frac{A}{4\pi t} + \frac{P_N - P_D}{8\sqrt{\pi t}} + \frac{1}{16}(C_{DD} + C_{NN} - C_{DN}) + O(t^\infty), \tag{10}$$

where A is the rectangle's area, P_N and P_D are the portions of its perimeter equipped respectively with Neumann and Dirichlet conditions, C_{DD} and C_{NN} are the numbers of corners where two Dirichlet or two Neumann sides meet, and C_{DN} is the number of corners where two sides of opposite type meet. Let us assume that (10) remains true for the remaining three cases. (Better yet, suppose that someone has proved that a formula of that *form*, but with unknown numerical coefficients, must hold for all rectangles. Then the three cases we have calculated suffice to determine the six coefficients.) We can then conclude that

$$K_{MM}(t) = \frac{L_1 L_2}{4\pi t}, \quad K_{DM}(t) = \frac{L_1 L_2}{4\pi t} - \frac{L_2}{4\sqrt{\pi t}}, \quad K_{NM}(t) = \frac{L_1 L_2}{4\pi t} + \frac{L_2}{4\sqrt{\pi t}} \tag{11}$$

without actually doing the calculations. In fact, we can reverse-engineer (11) to learn that $K_M(t) = L/\sqrt{4\pi t} + O(t^\infty)$ for the one-dimensional problem with mixed endpoints, (3).

As an inverse problem, suppose we listen to a rectangular drum closely enough to determine the small-t expansion of (7). From the leading term we learn the area of the drum. If there are no other terms, we know that the drum must be of the MM type. With some algebra the reader can see that in all other cases one can determine the type of the boundary conditions and the lengths L_1 and L_2.

The subject of the two books under review is the modern development of these and related ideas for very general classes of differential operators. The eigenfunctions ϕ are now sections of a vector bundle over a manifold, M, of arbitrary dimension, with or without boundary. Usually M is assumed compact, to make the spectrum discrete. Usually (but not always) the elliptic partial differential operator P (that takes over the role of $-\nabla^2$ in the rectangle example) is of second order and of "Laplace type" (meaning that the vectorial character of the functions does not enter the second-order terms). In those circumstances one can "complete the square" in the coordinate representation of P to express the derivative terms in terms of a Riemannian metric on M and a connection on the bundle: $P = -g^{jk}\nabla_j\nabla_k + 0\text{th-order term}$. If the concepts of metric and connection did not exist independently, they would be invented in an attempt to put a generic differential operator into a normal form by changes of variables.

In this setting one can *prove* a theorem corresponding to the assumption (10) of our toy exposition: The heat trace

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_M K(t,x,x)\,dx \tag{12}$$

has an asymptotic expansion in which the quantity multiplying each power of $\sqrt{t}$ is a universal linear combination of a finite (in each order) list of geometrically invariant quantities formed out of the metric, the curvature of the connection, the 0th-order term in P, and the functions defining the boundary conditions and the geometry of the boundary hypersurface. (For a precise statement, see Theorem 1.4.5 of Gilkey's book.) The explicit formula needs to be ground out order by order. (For a summary of the results to date, see Sec. 4.10 of Kirsten's book.) A systematic way of doing so is provided by the calculus of pseudodifferential operators developed by R. Seeley and others. P. Gilkey was a pioneer in the application of this method, and his lecture notes [2] provided a generation of mathematicians and physicists with a readable introduction to both pseudodifferential operators and index theorems. He quickly decided, however, that the calculational complexity of the systematic methods is intolerable in practice. Instead, he has concentrated on finding the numerical coefficients in the heat-kernel terms by more efficient and clever methods. The two principal tools were demonstrated in the rectangle example: *functorial relations* (e.g., the heat kernel of a Cartesian product is the product of those of the factors) and *special-case calculations* (solving enough cases to determine all the coefficients).

The contributions of the boundary conditions are especially complicated, and in recent years Gilkey has devoted much attention to them, in collaboration with T. Branson, M. van den Berg, K. Kirsten, and others. A related recent development is the study of *heat content* asymptotics, which concentrates on matrix elements of the type

$$\int_M \int_M \rho(x) K(t,x,y) f(y)\,dx\,dy \tag{13}$$

in place of the trace (12). (The name comes from the physical interpretation of f as the initial temperature distribution and ρ as the specific heat of the body.) The internal heat content series is rather trivial, so all the interest is attached to the boundary. Applications (mathematical or physical) of the content asymptotics are not as well developed as those of the trace asymptotics; however, an application to the free energy associated with a bounding surface of an electrolytic solution is developed in [5].

One has learned to expect a book from Peter Gilkey about heat asymptotics once every decade: [2] was followed by a more complete and polished volume [3], significantly revised ten years later, that emphasizes the topological and geometrical implications. The most recent book from Gilkey, reviewed here, is intensely focused. The emphasis is entirely on the calculation of heat expansions by functorial and special-case methods. The analytical underpinnings (needed to prove existence of the series) and the geometrical applications are covered only cursorily this time around. The book consists of three long chapters: one providing the needed geometrical and PDE background, the others on heat content and heat trace asymptotics. Perhaps the most original feature of the book is the decision to cover the heat content asymptotics first, which is arguably the more logical order, although not the historical one.

The slightly earlier book by Gilkey's frequent collaborator, Klaus Kirsten, covers much of the same ground but has a broader scope. The author specifically intends it for graduate students (of either mathematics or physics) who wish to begin serious research in the area. Several of its ten chapters are devoted to physical applications, such as vacuum energy and Bose condensation. Often these require knowledge of the entire heat kernel, not just its short-time behavior. One of Kirsten's specialties is finding exact forms for heat kernels and their Mellin transforms, the zeta functions, for systems with lots of symmetry. In particular, he and collaborators have worked hard on the "generalized cones" with metrics $ds^2 = dr^2 + r^2\,d\Sigma^2$, $d\Sigma^2$ being the metric on a lower-dimensional manifold. The analysis of the zeta function on the cone is reduced to that on the base manifold. Zeta function calculations are particularly effective in situations where the eigenfunctions can be expressed in terms of standard special functions, but no explicit formula for the eigenvalues is available; the analyticity of the zeta function can be used to convert sums over eigenvalues to contour integrals. Kirsten's book has a chapter on heat content asymptotics, but in comparison to Gilkey's it seems like an afterthought, an obligation to summarize one more thing that people are doing in the literature. Kirsten's book is well organized for reference, with several useful appendices in addition to the aforementioned summary section on the heat trace asymptotics.

Both books have lengthy and valuable bibliographies, containing approximately 400 items each.

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S. A. Fulling

Texas A&M University

E-mail address: fulling@math.tamu.edu