Metric spaces, convexity, and nonpositive curvature, by Athanase Papadopoulos, European Math. Soc., Zürich, 2005, xii + 287 pp., EUR 48.00, ISBN 3-03719-010-8

Let us begin by recalling a few basic concepts. A *metric space* is a nonempty set M together with a nonnegative real-valued distance function $d(x, y)$ defined for $x, y \in M$ such that $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ for every $x, y \in M$, and

$$(1) \qquad d(x, z) \leq d(x, y) + d(y, z)$$

for every $x, y, z \in M$. A fundamental example of a metric space is the real line $\mathbf{R}$ with the standard metric $|x - y|$, where $|x|$ is the *absolute value* of $x \in \mathbf{R}$, equal to x when $x \geq 0$ and to $-x$ when $x \leq 0$. More generally, if V is a real vector space, then a *norm* on V is a nonnegative real-valued function $N(v)$ such that $N(v) = 0$ if and only if $v = 0$, $N(\alpha v) = |\alpha| N(v)$ for every $\alpha \in \mathbf{R}$ and $v \in V$, and

$$(2) \qquad N(v + w) \leq N(v) + N(w)$$

for every $v, w \in V$, in which event

$$(3) \qquad d(v, w) = N(v - w)$$

defines a metric on V. A set $E \subseteq V$ is said to be *convex* if

$$(4) \qquad t\, v + (1 - t)\, w \in E$$

for every $v, w \in E$ and real number t such that $0 \leq t \leq 1$. If $N(v)$ is a norm on V, then

$$(5) \qquad \{ v \in V : N(v) \leq 1 \}$$

is a convex set. Conversely, if $N(v)$ is a nonnegative real-valued function on V which satisfies the requirements of a norm except perhaps for the triangle inequality and if (5) is convex, then $N(v)$ satisfies the triangle inequality and is therefore a norm.

For example, fix a positive integer n, and let V be $\mathbf{R}^n$, the space of n-tuples of real numbers. Put

$$(6) \qquad \|v\|_p = \Big(\sum_{j=1}^{n} |v_j|^p \Big)^{1/p}$$

when $1 \leq p < \infty$ and

$$(7) \qquad \|v\|_\infty = \max(|v_1|, \ldots, |v_n|).$$

It is easy to see that $\|v\|_p$ satisfies all of the conditions for a norm except perhaps the triangle inequality when $1 \leq p \leq \infty$, and that the triangle inequality holds when $p = 1, \infty$. For $1 < p < \infty$ one can use the convexity of the function t^p on the nonnegative real numbers to show that (5) is convex and hence that $\|v\|_p$ defines a norm on $\mathbf{R}^n$. Of course $\|v\|_2$ is the standard Euclidean norm on $\mathbf{R}^n$.

Let $\mathbf{Q}$ be the rational numbers. For every prime number p, the *p-adic absolute value* of $x \in \mathbf{Q}$ is denoted $|x|_p$ and defined to be 0 when $x = 0$, and to be p^{-l} when

$$(8) \qquad\qquad x = p^l \frac{m}{n},$$

where m, n are nonzero integers which are not divisible by p. One can check that

$$(9) \qquad\qquad |x + y|_p \leq \max(|x|_p, |y|_p)$$

and

$$(10) \qquad\qquad |x\,y|_p = |x|_p\,|y|_p$$

for every $x, y \in \mathbf{Q}$. The first property basically says that if x, y each have at least some number of factors of p, then $x + y$ does too, while the second property says that the total number of factors of p and x times y is the sum of the number of factors of p in x and y. The *p-adic metric* on $\mathbf{Q}$ is defined by

$$(11) \qquad\qquad d_p(x, y) = |x - y|_p,$$

and the *p-adic numbers* $\mathbf{Q}_p$ are the completion of the rational numbers with respect to the p-adic metric, in the same way that the real numbers are the completion of the rational numbers with respect to the standard metric.

An *ultrametric space* is a metric space $(M, d(x, y))$ in which

$$(12) \qquad\qquad d(x, z) \leq \max(d(x, y), d(y, z))$$

for every $x, y, z \in M$. The p-adic metric on $\mathbf{Q}$ is an ultrametric, and $\mathbf{Q}_p$ is an ultrametric space too. For any nonempty set A, let $\Sigma(A)$ be the space of sequences $x = \{x_j\}_{j=1}^{\infty}$ such that $x_j \in A$ for every $j \geq 1$, and for $x, y \in \Sigma(A)$ and $0 < \rho < 1$, put $d_\rho(x, y) = 0$ when $x = y$ and

$$(13) \qquad\qquad d_\rho(x, y) = \rho^l$$

when $x \neq y$ and l is the largest nonnegative integer such that $x_j = y_j$ for $j \leq l$. One can check that this defines an ultrametric on $\Sigma(A)$ for which the associated topology is the product topology, using the discrete topology on A. In general, ultrametric spaces are totally disconnected, which is to say that they contain no connected subsets with at least two elements, and specifically one can check that open balls are closed sets and closed balls are open sets.

If r, t are nonnegative real numbers and $0 < \alpha < 1$, then

$$(14) \qquad\qquad (r + t)^\alpha \leq r^\alpha + t^\alpha.$$

For instance, $\max(r, t) \leq (r^\alpha + t^\alpha)^{1/\alpha}$, and hence

$$(15) \qquad r + t \leq \max(r, t)^{1-\alpha}\,(r^\alpha + t^\alpha) \leq (r^\alpha + t^\alpha)^{1/\alpha}.$$

Consequently, for each metric space $(M, d(x, y))$, $d(x, y)^\alpha$ defines a metric on M when $0 < \alpha < 1$ which determines the same topology on M, sometimes described as a *snowflake transform* of M. If $(M, d(x, y))$ is an ultrametric space, then $d(x, y)^\alpha$ defines an ultrametric on M for every $\alpha > 0$ which determines the same topology on M. For $r, t > 0$ and $0 < \alpha < 1$, the inequality in (14) is strict, and hence

$$(16) \qquad\qquad d(x, z)^\alpha < d(x, y)^\alpha + d(y, z)^\alpha$$

for distinct elements x, y, z of M.

If $(M, d(x, y))$ is a metric space, $x, y, z \in M$, and

$$(17) \qquad\qquad d(x, z) = d(x, y) + d(y, z),$$

then y is said to be *between* x and z in the sense of Menger. In the case of the real line with the standard metric, this is equivalent to $x \leq y \leq z$ or $z \leq y \leq x$. If M is a real vector space equipped with a metric associated to a norm and if $x, z \in M$ and $0 \leq t \leq 1$, then $y = t\,x + (1 - t)\,z$ is between x and z. In an ultrametric space, the only way that y can be between x and z is if $y = x$ or $y = z$. The same holds in a snowflake space, although less dramatically.

If $a, b \in \mathbf{R}$, $a \leq b$, and $p(t)$, $a \leq t \leq b$, is a continuous path in the metric space $(M, d(x, y))$, then the (possibly infinite) *length* of this path is defined to be the supremum of

$$(18) \qquad\qquad \sum_{j=0}^{n} d(p(t_j), p(t_{j-1}))$$

over all partitions

$$(19) \qquad\qquad a = t_0 < t_1 < \cdots < t_n = b$$

of the interval $[a, b]$. The triangle inequality implies that the length of the path is greater than or equal to $d(p(a), p(b))$, and when equality holds, $p(t)$ is between $p(a)$, $p(b)$ for every $t \in [a, b]$. A line segment in a normed vector space has length equal to the distance between its endpoints. In a totally disconnected space, every continuous path is constant, while in snowflake spaces there can be plenty of continuous paths, but paths with finite length are constant because they have length 0 with respect to the metric from which the snowflake metric is obtained.

Papadopoulos' delightful book deals extensively with geometry related to betweenness and geodesics, including uniqueness of geodesics; i.e., is there exactly one curve connecting a pair of points whose length is equal to the distance between the points, up to reparameterization? For a vector space with a norm, uniqueness of geodesics corresponds to strict convexity. For example, on $\mathbf{R}^n$, $n \geq 2$, with the norm $\|v\|_p$, geodesics are unique when $1 < p < \infty$ and not in general when $p = 1, \infty$. Convexity of distances between geodesics is a quantitative condition that implies uniqueness of geodesics and is a form of curvature ≤ 0, as in "Busemann spaces". Local versions of these notions are also discussed.

Papadopoulos does a fantastic job of bringing together all sorts of themes in geometry, from introductory material for beginners to intricate properties of moduli spaces of Riemann surfaces. At times the treatment is necessarily somewhat sketchy, but still one can get some ideas, and references are given with additional information. The main body of the text is quite systematic, with digressions, examples, and notes in various directions, and should prove a valuable resource for students in particular. The historical comments are fascinating.

Papadopoulos has obviously put a lot of work into the book, and one cannot help but ask for more, like an expanded second edition. For example, it would seem natural to mention invariant metrics on cones, as in [21]. Nilpotent Lie groups have invariant geometries like Euclidean spaces in many ways, and also subtle differences which lead to a lot of interesting questions. One can consider sub-Riemannian spaces more generally, in which tangent vectors of curves are restricted to special subspaces in the tangent bundle. In addition to distances, one can take measures of subsets into account.

It seems to be one of life's little ironies that various kinds of singularities seem to show up even when they might not be expected. One might start with some nice smooth space with negative curvature, for instance, and be led to self-similar

structure on an asymptotic boundary. There are plenty of local versions of this too. Fortunately, surprising forms of regularity also frequently appear. Papadopoulos' book is a treasure trove of ideas and examples that many may enjoy.

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