BOOK REVIEWS

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Topology of singular spaces and constructible sheaves, by Jörg Schürmann, Birkhäuser-Verlag, Basel, 2003, x+452 pp., US$139.00, ISBN 3-7643-2189-X

The topology of singular spaces has been one of the main concerns of René Thom. Consequently, one should not be surprised to find his name early in the first chapter of the book (Thom-Sebastiani Theorem for constructible sheaves) and, all along the book, through the notions he introduced (stratification, regularity condition, isotopy lemmas,...). Strangely enough, no reference to Thom is made in the bibliographical section. This reveals two things: on the one hand, these notions are now so classical, so basic in any training on singular spaces, that it is no longer necessary to mention the sources; on the other hand, I would guess that the approach developed in this book would not have been much appreciated by Thom himself. Let me explain why in the example of the Sebastiani-Thom theorem. Here is the original conjecture,[1] which I cannot resist reproducing in its original language:

Soit $F(x_1, x_2, \ldots, x_i, \ldots, x_n)$ un germe de function analytique (resp. formelle) à l'origine O sur $\mathbb{C}^n(x)$ ou éventuellement $\mathbb{R}^n(x)$. On dira que le germe F est décomposable si, par un changement de variables local en O : $(x_i) \to (y_j)$, F peut s'écrire sous la forme d'une somme : $F = G(y_1, y_2, \ldots, y_s) + H(y_{s+1}, y_{s+2}, \ldots, y_n)$, G et H étant des germes de functions (analytiques ou formelles, sur $\mathbb{C}^s$ resp. $\mathbb{C}^{n-s}$, ou $\mathbb{R}^s$ resp. $\mathbb{R}^{n-s}$, de mêmes dimensions). En itérant sur chacun des termes G, H le même processus de décomposition, et en continuant sur chacun des termes obtenus tant que faire se peut, on finira par aboutir à une somme $g_1 + g_2 + \cdots + g_k$ de séries, dont chacune n'admet aucune décomposition. On dira alors qu'une telle somme est irréductible. Énonçons dès lors la « Conjecture » : (D) Deux décompositions irréductibles de la série F sont isomorphes (via un changement de variables local). [...] Il y a évidemment une connexion entre (D) et le théorème dit de Sebastiani-Thom : si (D) est vraie, il s'ensuit des contraintes très sévères sur les monodromies des singularités complexes correspondantes.... La théorie des catastrophes offre à ce problème une certaine motivation « philosophique ». Si l'on admet qu'un système physique voit son individuation créée par un cratère de potentiel $V(x_i)$ au voisinage de l'origine O, alors ce problème revient à reconnaître si un système physique donné n'est pas en fait le produit direct de deux systèmes indépendants non couplés. C'est

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[1] Although it probably goes back to the early seventies, it appears in published form only in 1989 [5].

donc le caractère distinguable (ou non distinguable) des entités qui est en jeu ici, ainsi que le caractère canonique des produits de systèmes indépendants.[2]

It does not seem that Conjecture (D) has been seriously considered up to now. On the other hand, many generalizations of the Sebastiani-Thom theorem have been obtained: they consist of exhibiting topological properties of the function $F(y_1, \dots, y_n) = G(y_1, y_2, \dots, y_s) + H(y_{s+1}, y_{s+2}, \dots, y_n)$ from analogous properties of the components G and H. Topologically, they correspond to the "join" construction (the join of two topological spaces is the space obtained by taking the union of all segments of length 1 from any point of the first one to any point of the second one). The first chapter of the book offers a very general sheaf-theoretic variant of it. I guess that Thom would have been more interested by advances in Conjecture (D) from a dynamical system point of view.

Which kind of singular spaces are considered in the book? Firstly, it is a question of sets defined by polynomial equalities or inequalities in $\mathbb{R}^n$ (real semi-algebraic sets). More generally, they are the sets locally defined by real analytic equalities or inequalities in $\mathbb{R}^n$, and lastly, whatever can be constructed from such a family of sets by taking projection on a linear subspace, finite intersection, and continuing like that. This process can be made abstract and gives rise to *o-minimal structures*.

The sets constructed with respect to such a procedure are *stratifiable*, that is, a union of disjoint pieces homeomorphic to open sets of an affine space or, more generally, to some topological manifolds, and they fulfill suitable incidence conditions near their boundary points. Although singular, these sets nevertheless keep some regularity properties, which distinguishes them from singular sets which usually appear in dynamical systems. If the incidence conditions are strong enough (Whitney conditions for instance), the topology of these sets is "locally finite": they locally have the homotopy type of a finite CW-complex.

Up to what point can one generalize to these singular sets the basic statements concerning the topological properties of manifolds? How can one define numbers which measure (from a topological point of view) the singularity of such a set? These questions have been asked in various ways.

Using Morse Theory, R. Thom ([1, 2]) was able to give bounds for the topology of those complex algebraic subsets of $\mathbb{C}^n$ which are manifolds (affine complex manifolds): they are finite CW-complexes consisting of cells of dimension less than or equal to the complex dimension of the affine manifold. These techniques have

[2]Let $F(x_1, x_2, \dots, x_i, \dots, x_n)$ be a germ of analytic (resp. formal) function at the origin O on $\mathbb{C}^n(x)$ or possibly $\mathbb{R}^n(x)$. One will say that the germ F is decomposable if, by some local variable change at O: $(x_i) \to (y_j)$, F can be written as a sum $F = G(y_1, y_2, \dots, y_s) + H(y_{s+1}, y_{s+2}, \dots, y_n)$, G and H being germs of functions (analytic or formal, on $\mathbb{C}^s$ resp. $\mathbb{C}^{n-s}$, or $\mathbb{R}^s$ resp. $\mathbb{R}^{n-s}$, with the same dimensions). By iterating on each term G, H the same decomposition process and going through on each term as far as possible, one will end up with a sum $g_1 + g_2 + \cdots + g_k$ of series, none of which has a decomposition. One will then say that such a sum is irreducible. Let us then state the "Conjecture": (D) Two irreducible decompositions of the series F are isomorphic (through a local change of variable). [...] There is clearly a connection between (D) and the so-called Sebastiani-Thom theorem: if (D) is true, this implies severe constraints on the monodromy of the corresponding complex singularities.... Catastrophe Theory offers some "philosophical" motivation to this problem. If one admits that a physical system has its individuation created by a potential crater $V(x_i)$ in some neighbourhood of the origin O, then this problem amounts to recognizing whether a given physical system is the product of two independent, not coupled, systems. It is the distinguishable (or not distinguishable) character of entities that is concerned here, as well as the canonical character of the products of independent systems.

been extended to very general situations (in the framework mentioned above) by H. Hamm and D. T. Lê on the one hand and by M. Goresky and R. D. MacPherson [3] on the other hand: for instance they provide bounds on the topology of complex algebraic subsets of $\mathbb{C}^n$ in terms of their singularities.

The author mainly considers singular spaces from a cohomological point of view. In this way, many fine topological phenomena are not visible. Nevertheless, he not only uses constant coefficients but also twisted coefficients, as well as the stratified singular version of them, that is, constructible sheaves, so that the theory is still very rich. In the case of complex spaces, it is now known that one can superimpose on the theory developed in the book a finer structure, coming from Hodge theory, and that the very abstract formalism which is used by the author has the advantage of falling in with the introduction of Hodge theory. It also appears in arithmetic algebraic geometry (ℓ-adic sheaves, for instance). It therefore possesses a wide universality.

As mentioned on the back cover of the book, the reader is assumed to be familiar with sheaf theory. I would say *more* than familiar. A very good knowledge of the contents of the reference book [4] is really assumed. The book can be regarded as a complementary reading to [4], with more emphasis on geometric situations. Indeed, many links are provided by the author between [4] and more geometric or topological approaches, as that of Goresky and MacPherson [3]. He takes for granted one of the essential notions introduced, after M. Sato, by M. Kashiwara and P. Schapira, namely that of microlocalization of sheaves. That sheaf theory can benefit from the advances in analysis leads us back to the first chapter concerning the Sebastiani-Thom theorem: this result can be regarded as a sheaf-theoretic analogue of a classical result of analysis saying, when meaningful, that the Fourier transform exchanges the ordinary product of functions with the convolution product.

Now for a quick description of the book. It begins somewhat abruptly with a quite technical chapter on the Sebastiani-Thom theorem. In my opinion, this chapter could have been the last one in order not to frighten the beginner. Chapter 2 is of a more introductory nature and expounds sheaf theory in various situations (complex algebraic, real algebraic, real subanalytic, o-minimal structures). After an adaptation of the classical localization results in equivariant cohomology to this very general framework, the author develops, in Chapters 4–6, Morse theory for constructible sheaves, here also in various general contexts, and makes the link between the previous approaches of Goresky-MacPherson and Kashiwara-Schapira.

Each chapter begins with a long, very complete and motivating introduction, which could suffice in a first reading and which makes the book very attractive.

References

[1] A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. (2) **69** (1959), 713–717. MR0177422 (31:1685)

[2] R. Bott, *On a theorem of Lefschetz*, Michigan Math. J. **6** (1959), 211–216. MR0215323 (35:6164)

[3] M. Goresky and R. D. MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 3 Band 14, Springer-Verlag, Berlin, Heidelberg, New York, 1988. MR0932724 (90d:57039)

[4] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer-Verlag, 1990. MR1074006 (92a:58132)

[5] R. Thom, *Problèmes rencontrés dans mon parcours mathématique: un bilan*, Publ. Math. Inst. Hautes Études Sci. (1989), 199–214; see also [6]. MR1067383 (92a:58001)
[6] ———, *Œuvres complètes*, Institut des Hautes Études Scientifiques, Bures-sur-Yvette, 2002, on a CD-Rom.

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