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Ramanujan's lost notebook, Part I, by George E. Andrews and Bruce C. Berndt, Springer, New York, 2005, xiv+437 pp., US$89.95, ISBN 0-387-25529-X

Ramanujan's story is one of the great romantic tales of mathematics. It is an account of triumph and tragedy, of a man of genius who prevailed against incredible adversity and whose life was cut short at the height of his powers. The extent of those powers is only now being fully recognized. Ramanujan had the misfortune to work on problems that, in his time, were considered a mathematical backwater. Modular equations, theta function identities, even continued fractions were viewed as having been played out in the nineteenth century. One might pick up tidbits, but there was nothing important left to be discovered.

G.H. Hardy knew the error of this view. In his twelve lectures given at Harvard in 1936 [31], he communicated the range and depth of Ramanujan's work. Their asymptotic series for the number of partitions of an integer, published in 1918 [32], later refined by Rademacher [35] into a rapidly convergent series, remains one of the great achievements of analytic number theory. Hardy credited Ramanujan for all of the inspiration. Nevertheless, even Hardy expressed uncertainty about the true greatness of Ramanujan's accomplishments:

> Opinions may differ about the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work. [31, p. 7]

To the uninitiated, it is difficult to assess the significance of Ramanujan's formulas (see table 1). Equations (1.1) and (1.2) are from Ramanujan's introductory letter to Hardy. Equation (1.1) was one Hardy could prove, "though with a good deal more trouble than I had expected." Equation (1.2) mystified him, convincing him that Ramanujan must be "a mathematician of the highest class," for no one could have had the imagination to invent such an identity, and "great mathematicians are commoner than thieves or humbugs of such incredible skill" [31, p. 9].

1. Mock theta functions

While Ramanujan's work includes a variety of problems in number theory and the evaluation of series and integrals, he was most at home in the world of modular forms. The variable q stands in for $e^{2\pi i\tau}$, Im $\tau > 0$, and much of his inspiration came from classical results for theta functions and elliptic functions. He sought to extend what was known about these functions to broader classes of q-series. In the last year of his life, following his return to India in 1919, he explored what he named "mock theta functions".

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(1.1)
$$\int_0^\infty \prod_{j=1}^{\infty} \frac{1+(x/(b+j))^2}{1+(x/(a+j-1))^2}\,dx \quad = \quad \frac{\pi^{1/2}}{2}\cdot\frac{\Gamma(a+1/2)\Gamma(b+1)\Gamma(b-a+1/2)}{\Gamma(a)\Gamma(b+1/2)\Gamma(b-a+1)},$$

(1.2)
$$\frac{1}{1+}\;\frac{e^{-2\pi}}{1+}\;\frac{e^{-4\pi}}{1+\cdots} \quad = \quad \left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)e^{2\pi/5},$$

(1.3)
$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} \;=\; \prod_{j=1}^{\infty}\frac{1}{(1-q^{5j-1})(1-q^{5j-4})},$$

(1.4)
$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)}$$
$$= 2+\prod_{j=1}^{\infty}\frac{(1-q^{5j})(1-q^{10j-5})}{(1-q^{5j-4})(1-q^{5j-1})}$$
$$-2\sum_{n=0}^{\infty}\frac{q^{10n^2}}{(1-q^2)(1-q^8)(1-q^{12})(1-q^{18})\cdots(1-q^{10n+2})}.$$

TABLE 1. Four of Ramanujan's identities.

To Ramanujan, a theta function is any sum of rational products of the classical theta functions: series of the form

$$\sum_{n=-\infty}^{\infty}(-1)^{\epsilon n}q^{an^2+bn}, \quad |q|<1,$$

where a is a positive integer, b is an integer, and $\epsilon = 0$ or 1. Ramanujan made great use of the fact that theta functions have simple asymptotic behavior in terms of t, $q = e^{-t}$, as $t \to 0$. He focused on what he called "Eulerian series", really basic hypergeometric series, in which the ratio of the $n+1^{\text{st}}$ summand to the n^{th} is a rational function of q^n. The left sides of equations (1.3) and (1.4) are Eulerian series. The right side of equation (1.3), the famous first Rogers-Ramanujan identity, is a theta function. Therefore, this series has a simple asymptotic formula. A mock theta function is an Eulerian series that is not a theta function but for which, in Hardy's words, "we can calculate asymptotic formulæ, when q tends to a 'rational point' $e^{2r\pi i/s}$, of the same degree of precision as those furnished, for the ordinary θ-functions, by the theory of linear tranformation" [36, p. 354]. The difficulty is not in finding the asymptotics. Though not trivial, the method is based on the Hardy-Ramanujan-Rademacher expansion and is well understood. The real difficulty comes in proving that the Eulerian series in question is not a theta function. In fact, this has never been accomplished, but equation (1.4) strongly suggests that the left side is, indeed, a *mock* theta function.

The real significance of Ramanujan's identities often requires some digging. Equation (1.4) is related to partition congruences. Ramanujan observed and proved that the number of partitions of $5n+4$ is divisible by 5, of $7n+5$ is divisible by 7, and of $11n+6$ is divisible by 11. Freeman Dyson, seeking natural equinumerous subsets of these partitions, conjectured that the congruence class of the rank, the largest part minus the number of parts, provides such a division for the moduli 5 and 7. This was proven by A.O.L. Atkin and P. Swinnerton-Dyer [20]. Equation (1.4) yields a simpler proof. It led Frank Garvan [30] to his discovery of the "crank", a partition parameter that unifies the combinatorial explanation of the Ramanujan congruences for the moduli 5, 7, and 11.

2. Discovery and context of the lost notebook

The designation "lost notebook" is doubly incorrect. It is not a notebook, rather a collection of loose pages. And it had not been lost. R.A. Rankin and J.M. Whittaker knew of its existence, and it has always been archived as Ramanujan papers in the Trinity College Library in Cambridge, where it resides.

Most of Ramanujan's work was unpublished at the time of his death. It lay in three notebooks generally believed to predate his voyage to England in 1914, in the lost notebook consisting primarily of research that followed his return to India, and in scattered additional pages that today are included as part of the lost notebook. After Ramanujan's death, the papers containing his work on mock theta functions and much else were collected and shipped to England, at some point passing through Hardy's hands. They resurfaced among the papers of G.N. Watson at his death in 1965. Though recognized as Ramanujan's writing, their true significance was not appreciated.

Until 1976, knowledge of Ramanujan's work on mock theta functions came solely from a cryptic letter he had sent to Hardy early in 1920. Still, this was enough for Watson to publish results on the subject in the 1930s [39, 40]. In the papers, Watson appears to be ignorant of the content of the lost notebook, suggesting that it was not yet in his possession. In 1964 George Andrews, writing his doctoral dissertation [2] under the direction of Rademacher, found the explicit asymptotic formulas for the mock theta functions treated by Watson. Thus, Andrews was well prepared to recognize the importance of Ramanujan's papers when he first saw them during a trip to England in 1976. While mock theta function results constitute only about 5% of the manuscript, they are scattered throughout, confirming that this represents Ramanujan's final work.

Andrews' discovery of the lost notebook came amid a series of events that was heightening interest in Ramanujan's work and in q-series in general. Andrews' trip to England came at the end of a year of collaborative work with Richard Askey in Madison, a year in which Andrews had completed the manuscript of *The Theory of Partitions* [3], exploring the role of Ramanujan's identities as partition-generating functions, and a year in which he and Askey had begun to explore the rich interplay between orthogonal polynomials and q-series. This is a subject with a long history going back to E. Heine, L.J. Rogers, and F.H. Jackson, but whose potential of interesting and important questions had been largely untapped.

Two years earlier, Bruce Berndt had learned of Ramanujan's unpublished work on modular equations within the three original notebooks. In 1977, working from the Tata Institute's facsimile publication of Ramanujan's pre-1914 notebooks [37], he began the systematic study of the identities in chapter 14. Watson had worked with B.M. Wilson in the 1930s, beginning the task of cataloging, proving, and putting into context the results of Ramanujan's first three notebooks. Wilson died in 1935. Watson published over 30 papers based on Ramanujan's identities, but he never completed the systematic treatment of the notebooks. Berndt learned from Andrews that the notes from the Watson and Wilson efforts were also archived at the Trinity College Library. The editing of chapter 14 turned into the editing of the entire contents of all three notebooks, a project that has inspired decades of important mathematics and which was not completed until 1998 [21, 22, 23, 24, 25]. It continues now in this new series on the lost notebook.

Also in the late 1970s, Rodney Baxter, working on the hard hexagon model of statistical mechanics, rediscovered Ramanujan's equation (1.3). He contacted Andrews, and a fruitful collaboration ensued [5, 14, 15, 16]. Their work, building on Ramanujan's insights into q-series, has continued under the attention of such physicists as Alexander Berkovich, Peter J. Forrester, Barry M. McCoy, Anne Schilling, and S. Ole Warnaar.

Another connection to physics, in this case to quantum physics and string theory, also began to develop in the 1970s. In his 1972 Gibbs Lecture [28], Freeman Dyson explained the relationship between Ramanujan's eta-function identities and I.G. Macdonald's work on affine or Kac-Moody Lie algebras. Later that decade, James Lepowsky and Steven Milne [33] deepened this connection, setting the stage for much of the subsequent work on affine Lie algebras.

Closing the decade of the '70s, Macdonald produced his book *Symmetric Functions and Hall Polynomials* [34]. Representation theory has always been intimately tied to partition theory. The earliest mention of the partition counting function occurs in a letter from G.W. Leibniz to Johann Bernoulli. Leibniz observed that, in modern terminology, the monomial symmetric functions of degree n in k variables are naturally indexed by partitions of n into at most k parts. He asked Bernoulli whether he knew of an expression for the number of such partitions. Macdonald's book, while building on a strong tradition that runs through I.J. Schur, D.E. Littlewood, and many others, marks the point at which the true depth and complexity of the relationship between representation theory and q-series was first realized.

In the space of a few years, the study of Ramanujan's work had moved from a quiet garden—borrowing Freeman Dyson's metaphor [29]—tended by a few devotees, to a busy horticultural center in which exotic species were cross-pollinating at a prolific rate. Important and easily accessible problems abounded. One indication of Ramanujan's impact is the success and importance of *The Ramanujan Journal*, founded in 1997 by Krishnaswami Alladi to "publish original research papers of the highest quality in all areas of mathematics influenced by Srinivasa Ramanujan."

While much, perhaps even most, of the work on q-series has not been directly informed by Ramanujan's results, they permeate the subject. Most of Ramanujan's work is so original, so unique, so far from what one expects, that one never knows where it will crop up and provide critical insight.

3. Impact of the lost notebook

The lost notebook contains approximately 650 identities, most of which were previously unknown. Andrews has published two excellent overviews of the lost notebook: a *Monthly* article [4] and the introduction to the facsimile publication of the lost notebook and other Ramanujan papers [38] that was produced to commemorate the centenary of Ramanujan's birth in 1987. One of the most surprising aspects of this notebook is the variety of results that Ramanujan discovered. There are continued fraction identities, transformations for q-series, identities for partial and false theta functions (essentially classical theta functions with truncated summations), q-series identities relevant to partitions and partition congruences, identities for mock theta functions, integral evaluations, modular equations and relations, and theta type series involving indefinite quadratic forms.

This last category of results has been particularly fruitful. One of the functions that fascinated Ramanujan is $R(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}/(1+q)(1+q^2)\cdots(1+q^n)$, a series that is related to Hecke modular forms. Andrews popularized this function in a *Monthly* article in 1986 [10], where he pointed out that, expanded as a power series, the coefficients exhibit a peculiar behavior. Andrews conjectured that the coefficients were not bounded but that nonzero coefficients were relatively sparse. This series drew interest from many mathematicians, eventually leading to a joint publication by Andrews, Dyson, and Hickerson [17], followed by a paper of Henri Cohen [27] that has raised new questions in algebraic number theory. The function $R(q)$ turns out to be one of two symmetric pieces of a q-analog of the Artin L-function for a character defined on the ideals of $\mathbb{Z}[\sqrt{6}]$.

Since 1976, Andrews, Berndt, their students, and others have published over a hundred and fifty articles based on Ramanujan's results from the lost notebook. One series of papers that is particularly noteworthy is Andrews' "Ramanujan's 'lost' notebook. I–IX" [6, 7, 8, 9, 11, 12, 13, 18, 19].

The volume under review is the first of what is anticipated to be four volumes that systematically explain, prove, and set into context every result from Ramanujan's lost notebook as well as his other unpublished papers. To the extent possible, the results are organized topically with cross-references to the identities as they appear in the original Ramanujan manuscript. Particularly helpful are the extensive references, indicating where in the literature these results have been proven or independently discovered as well as where and how they have been used.

This first volume begins with results on continued fractions. Even within this restricted domain, the variety is impressive. The first five chapters deal with the Rogers-Ramanujan continued fractions. Their evaluations, of which equation (1.2) is one example, are based on the connection to modular equations via the Rogers-Ramanujan identities. Identities for other q-continued fractions are found in chapter 6. Chapter 7 develops asymptotic formulas for continued fractions. Chapter 8 studies a continued fraction expansion of $\prod_{j=1}^{\infty}(1-q^{3j-1})/(1-q^{3j-2})$.

Chapters 9–11 deal with q-series: transformations that follow from the Rogers-Fine identity, empirical evidence for the Rogers-Ramanujan identities including comparison of asymptotics, and extensions to Rogers-Ramanujan-type identities for other moduli. Chapter 12 introduces identities based on partial fraction decomposition. Chapter 13 finds representations of q-series as Hadamard products. Chapters 14–16 present integrals of theta functions, incomplete elliptic integrals, and integrals of q-products. This volume ends with modular equations and identities for Lambert series.

To convey some sense of what can be found in this volume, let me share one result of Ahlgren, Berndt, Yee, and Zaharescu [1] that implies two of Ramanujan's integral evaluations, cases in which the integrand can be expressed as a rational product of eta-functions.

Theorem 3.1. *Suppose that α is real, $k \geq 2$ is an integer, and χ is a nontrivial Dirichlet character that satisfies $\chi(-1) = (-1)^k$. Then, for $0 < q < 1$,*

$$q^{\alpha} \prod_{n=1}^{\infty} (1 - q^n)^{\chi(n) n^{k-2}} = \exp \left(-C - \int_q^1 \left\{ \alpha - \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} t^n \right\} \frac{dt}{t} \right), \tag{3.1}$$

where $C = L'(2 - k, \chi)$.

Publication of these four volumes on the lost notebook will complete the task of making all of Ramanujan's mathematics easily accessible. This will facilitate the real work. It is one thing to be able to prove an identity, quite another to understand how to fit it into a comprehensive theory. Andrews, Berndt, and those who have worked with them have made substantial progress toward understanding the true significance of each of these identities, but I anticipate many pleasant surprises yet to be revealed.

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