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Selected papers of Peter Lax, Peter Sarnak and Andrew Majda (Editors), Springer, 2005, xx+620 pp., US$119.00, ISBN 0-387-22925-6

Peter Lax's mathematical work is a harmonious combination of prewar Budapest and postwar New York. By Budapest, I mean his elegant geometric, functional-analytic style of attacking hard, concrete problems. By New York or NYU, I mean his lifelong interest in physics and in practical computation. Like Peter's idol Johnny von Neumann (Neumann Jancsi at home), Peter's education benefited from the prewar Budapest institution of home tutoring by unemployed mathematical geniuses. For Johnny, it was Gabor Szegö and Michael Fekete; for Peter, it was Rozsa Peter. He said, "The very first thing we did was to read *The Enjoyment of Mathematics* by Rademacher and Toeplitz. I was twelve or thirteen. ...I went to her house twice a week for a period of a couple of years, until I left Hungary at fifteen and a half." The Laxes sailed from Lisbon on the fifth of December 1941, two days before the Japanese attacked Pearl Harbor and brought the United States into the war. When the Laxes arrived in New York, Richard Courant had been forewarned. "My Hungarian mentors were von Neumann and Szegö; it was Szegö who suggested that I study with Courant."

In 1986, in Tokyo en route to Palo Alto, I bumped into Pál Erdös. I told him I was on my way to Peter's 60th birthday party. Erdös said, "I knew him when he was 15." Peter said, "When I first met Erdös I was still at Stuyvesant; he invited me a number of times to visit him in Princeton at the Institute for Advanced Study. Once he introduced me to Einstein as a talented young Hungarian mathematician. Einstein was genuinely puzzled. 'Why Hungarian?'"

After a happy year at Stuyvesant High in Manhattan, Peter joined NYU, where he has remained happily ever since (with many interludes at Los Alamos, Stanford, and elsewhere). In a graduate class on complex variables he met his brilliant and beautiful fellow student and future bride, Anneli. After earning his B.A. and then his U.S. Army honorable discharge (as a corporal at the Los Alamos Manhattan Project), he wrote a thesis under Courant's famous student, Kurt Otto Friedrichs. It was, of course, on partial differential equations, which would be Peter's lifelong mathematical milieu. "Partial differential equations". Sounds "partial", doesn't it? Or, in contrast to "ordinary differential equations", extraordinary? weird? very specialized? Well, modern physics is relativity and quantum mechanics—the Einstein equations and the Schrödinger equation.

Classical physics is continuum mechanics, mainly fluid dynamics, which includes the atmosphere (meteorology) and the oceans (oceanography)—the Navier-Stokes equation. All particular, special examples of PDE's! But what about Newtonian mechanics? Well, that's an ODE, one of those special PDE's with only one independent variable. But the variational approach turns it into the Hamilton-Jacobi partial differential equations. Does that mean PDE equals physics? No, not quite.

Complex analysis is all about the Cauchy-Riemann equations. Differential geometry is all about the Weingarten and the Codazzi-Mainardi equations. Calculus of variations is all about the Euler-Lagrange equations.

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What about topology? Well, the hottest thing there right now seems to be the Hamilton-Perelman proof of the Poincaré-Thurston conjectures on 3-manifolds. That turns out to be all about a particular nonlinear parabolic equation for "Ricci flows".

What about algebra? Well, PDE's with constant coefficients are simply polynomial equations with differential operators as their variables. Consequently, the essence of that simplest part of PDE is the connection between algebraic conditions on the polynomial and analytic properties of the solutions; thus the classification, originally of second-order and, in modern times, of general constant coefficient PDE's into "elliptic", "parabolic", "hyperbolic", "Petrowsky-correct" and so on types. Here, rather than contribute solutions to algebra, PDE contributes problems, new and challenging ones.

One of my transformative experiences as a graduate student under Peter Lax was my consultation with Andre Weil in 1960. It was about a thesis problem I had innocently chosen from Peter's list of available problems. The question was hyperbolicity: i.e., how to represent a certain polynomial all of whose roots are real. Fortunately, Weil had come up to New York from Princeton and could be approached in the lounge on the 13th floor of the Courant Institute. Prof. Weil kindly permitted my question and answered very briefly, "Such questions about real roots are very difficult." (After a few months I switched to an easier problem.) Now I learn, on page 587 of Volume II of this book, that Peter's conjecture was recently proved by A.S. Lewis et al. The proof hinged on an observation of Helton and Vinnikov, based on a deep result of Vinnikov on self-adjoint determinental representation of real plane curves which Vinnikov published in 1993. Thus Peter Lax's specialty in partial differential equations opened up all of mathematics and physics to his enjoyment.

The two volumes of the *Selecta* are divided into nine parts. In his instructions the editor advises against going through the book chapter by chapter. But in this case, looking at the parts of the book is a good way to appreciate the variety and depth of Lax's work. As to my duty to "give a sense of what is going on in the particular field represented by the book," I would hardly dare to attempt that, but the list of Peter's contributions may to some extent serve that purpose.

Part I is called "Partial Differential Equations". Perhaps the most often referenced paper here is [20] from Peter's CV, "Asymptotic solutions of oscillatory initial value problems". This rather concrete, classical paper on a seemingly rather special subject has ideas which reverberated over the following decades through general theories of "Fourier integral operator calculus", especially by Lars Hörmander and his epigones.

Part II is "Difference Approximations to PDE". As a general rule, if you want more than qualitative information about a PDE, even a linear one with variable coefficients and especially a nonlinear one, your only hope is to approximate it. Of course, the point is to approximate it with equations you *can* solve, by partial difference equations whose solutions can be "written down" recursively. Central here is the "Lax-Richtmyer theorem", which virtually revolutionized theoretical numerical analysis of partial differential equations of evolution (hyperbolic and parabolic), even though it's merely a direct application of the elements of functional analysis. People had assumed that determining whether a difference approximation to a PDE is "convergent" and whether it's "stable" constitute two different problems. Peter and Bob Richtmyer noticed and proved that they're actually the same problem.

In Part III, "Hyperbolic Systems of Conservation Laws", we find the "Lax-Wendroff" method, one of Peter's earliest and most popular contributions (with my friend Burt Wendroff of Los Alamos). This section is where we meet shock waves, discontinuous solutions of nonlinear hyperbolic equations. What can it mean to talk about a "discontinuous solution" to a differential equation, an equation about the derivatives of the unknown function? Well, there certainly are shocks—sonic booms, for example—in the atmosphere. So, mathematically, these must be discontinuous solutions of the relevant equations of fluid dynamics. Of course, making sense of this is a challenging mathematical problem. It turns out that the hard part isn't so much saying what you mean by the existence of such solutions, but finding the appropriate supplementary condition to make them unique (as they presumably should be, if fluid dynamics is really deterministic). Here Peter contributed his famous entropy condition. And again, amazingly, he contributed, in an important special case, actually a new explicit solution!

Part IV, "Integrable Systems", deals with "solitons". Solitons are solutions of nonlinear PDE's that (totally unexpectedly) turn out to preserve their form or shape after undergoing mutual collisions or interactions. More precisely, as Martin Kruskal and Norman Zabusky discovered, solitons of different sizes interact cleanly; i.e., as t tends to infinity each soliton preserves its shape. So in some sense they're like particles (hence the "—ons", to sound like bosons or fermions or hadrons). The discovery of "solitons" goes back to 1834, when John Scott Russell saw a solitary wave "some thirty feet long and a foot and a half in height" traveling at 8 or 9 miles an hour down the Union Canal at Hermiston, near Edinburgh, Scotland. (He chased it on horseback for one or two miles.) This has been modeled by the Korteweg-DeVries equation of fluid dynamics. It turns out that there is a large collection of other particular nonlinear PDE's that permit "solitons". Furthermore, it turns out (due to brilliant work of Cliff Gardner, J. Greene, Martin Kruskal and Bob Miura, recently honored by a Steele Prize) that these equations can be, in a new sense of the term, linearized, by the inverse scattering method, and thus related to—what?—nothing other than the linear Schrödinger equation of quantum mechanics. Here Lax's contribution was named, by other people, "the Lax pair". It's an operator equation, of very simple form, involving two noncommuting linear operators, that has become one of the few standard tools for analyzing the soliton phenomenon.

Parts V and VI are devoted to Peter's creation, in collaboration with Ralph Phillips, called the Lax-Phillips scattering theory. In quantum mechanics, scattering theory is Heisenberg's approach to the Schrödinger equation. It's simply an input-output approach to quantum-mechanical interactions. The scattering operator transforms the incoming wave into the outgoing wave. This general or abstract formulation applies just as well to classical as to quantum phenomena. In the hands of Lax and Phillips, it's a beautiful new special topic in functional analysis. But what is amazing, at least to me, are its deep and totally unexpected connections to number theory and non-Euclidean geometry! The applicability of the Lax-Phillips scattering theory to waves in non-Euclidean space was discovered by Ludwig Fadeev and Boris Pavlov, Peter reminds me. Peter said, "I like to start with some phenomenon, the more striking the better, and then use mathematics to try to understand it. The formation, propagation, interaction, decay of shock waves is an example of a striking phenomenon. So is the scattering of electromagnetic and acoustic waves.

So is the stability and coherence of solitary waves, governed by completely integrable equations. So is the weak but not strong convergence of oscillating solutions of dispersive systems. I find such problems both natural and central."

I omit comment on Part VII, a mixed bag on "Analysis", and Part VIII on "Functional Analysis". Part IX is "Algebra". It includes a description of the collaboration of Peter and Ralph Phillips with the famous algebraic topologist J. F. Adams.

These two volumes, edited by Peter Sarnak and Andy Majda, contain helpful remarks by several of Peter's many students and friends.

Perhaps the editor will permit this all-too-brief review to conclude with a personal comment. There are many versatile and profound mathematicians. Occasionally it has been mentioned that such mathematical standouts aren't necessarily kind or gentle human beings. The fact that Peter Lax actually is such a human being is, in my opinion, the most exceptional and admirable thing about him.

References

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