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Time-frequency and time-scale methods, by Jeffrey A. Hogan and Joseph D. Lakey, Birkhäuser, Boston, 2005, xxii+390 pp., US\$74.95, ISBN 0-8176-4276-5

Harmonic analysis, in a fundamental sense, may be understood as the analysis based on the decomposition of complicated mathematical objects into simple building blocks. The prime example of such a decomposition is a Fourier series, where a periodic function is expanded into its fundamental modes or pure frequencies. Likewise the Fourier transform of a function on $\mathbb{R}^d$, defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x}\, dx \,, \tag{1}$$

and the corresponding inversion formula analyze the frequency content of f.

Underlying many finer decompositions into simpler pieces are the fundamental invariance properties of $\mathbb{R}^d$ and the corresponding operators. For harmonic analysis on $\mathbb{R}^d$ these are the translations $t \to t + x$ and the dilations $t \to at$, where $t, x \in \mathbb{R}^d$ and $a \in \mathbb{R}$. The corresponding operators are the *translation operator* T_x

$$T_x f(t) = f(t - x) \qquad t, x \in \mathbb{R}^d \,, \tag{2}$$

and the *dilation operator* D_a

$$D_a f(t) = |a|^{-d/2} f(t/a) \,. \tag{3}$$

To achieve symmetry with respect to the Fourier transform, we add the *modulation* operators M_ξ:

$$M_\xi f(t) = e^{2\pi i \xi \cdot t} f(t) \qquad t, \xi \in \mathbb{R}^d \,. \tag{4}$$

Then we have $(M_\xi f)\widehat{\ }(\omega) = \hat{f}(\omega - \xi)$ and $(D_a f)\widehat{\ }(\omega) = D_{1/a}\hat{f}(\omega)$, so that the whole set of operators $T_x M_\xi D_a$ is invariant under the Fourier transform.

When the analysis is based on translations and modulations, we speak of time-frequency analysis or phase-space analysis; when translations and dilations are used, we speak of time-scale analysis. In a broader sense, the combination of all three operations (joint time-frequency time-scale methods) is often referred to as time-frequency analysis.

In the evolution of analysis, time-frequency and time-scale methods have experienced many reincarnations. Time-frequency analysis has its origins in the work of H. Weyl [12] and J. von Neumann [10] on the mathematical foundations of quantum mechanics, and of D. Gabor on information theory [5]. Their original problems were the formulation of uncertainty principles, coherent states expansions, and quantization via pseudodifferential operators. These topics are still an inspiration for current research, and indeed two chapters in this book are devoted to uncertainty principles and pseudodifferential operators. The classical version of time-scale analysis is the Littlewood-Paley theory for the decomposition of L^p-spaces and Hardy spaces and the theory of Calderòn-Zygmund operators [6].

The combined time-frequency time-scale analysis is much more mysterious. It is part of some of the deepest and most difficult theorems in harmonic analysis. For instance, the proof of Carleson's celebrated theorem on the almost everywhere

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convergence of Fourier series by Lacey and Thiele relies heavily on a subtle decomposition of a maximal operator, the Carleson operator, with respect to a set of time-frequency shifts and dilations of some function [9]. Likewise, combined time-frequency and time-scale methods occur in the solution of Calderòn's conjecture on the boundedness of the bilinear Hilbert transform by Lacey and Thiele [8]. Some of these ideas are briefly described in Chapter 7 of this book in the context of the so-called Walsh model.

This book is devoted to modern incarnations of time-frequency analysis and time-scale analysis. The evolution of the last 20 years has reshaped time-frequency and time-scale analysis. New topics and techniques have emerged in part through the intensive interaction with signal and image processing and other modern engineering applications. Classical ideas have resurfaced in applicable shape. Many of the modern aspects may be subsumed under the notions of dictionaries, transforms, redundancy, and adaptive methods.

(i) *Dictionaries*: In modern terminology, a set of building blocks is called a dictionary; all dictionaries in this book are subsets of the time-frequency time-scale dictionary $\mathcal{D}_{\max} = \{M_\xi T_x D_a g : x, \xi \in \mathbb{R}^d, a > 0\}$ of a single function g (or several functions g_j). A time-frequency time-scale shift of g is called a *wave packet*, and $\mathcal{D}_{\max}$ is the maximal wave packet dictionary.

At least two mathematical directions are defined explicitly by the study of certain structured subsets of $\mathcal{D}_{\max}$. The analysis of dictionaries containing only the time-frequency shifts $\mathcal{D}_G = \{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}^d\}$ is *Gabor analysis* [3], the study of dictionaries containing only translations and dilations $\mathcal{D}_W = \{D_{2^j} T_k g_r : k \in \mathbb{Z}^d, j \in \mathbb{Z}, r = 1, \dots, 2^d - 1\}$ defines *wavelet theory* [2].

(ii) *Transforms*: To any dictionary $\mathcal{D}$ we associate a transform $C_{\mathcal{D}}$ mapping functions into sequences indexed by $\mathcal{D}$ as follows:

$$C_{\mathcal{D}} f(d) = \langle f, d \rangle \qquad d \in \mathcal{D}\,, \tag{5}$$

where we identify the dictionary with the index set. Although this coefficient operator may be studied as an abstract mathematical object, the transforms associated to wave packet dictionaries carry concrete physical information about a function f. Using only time-frequency shifts, the transform is

$$\langle f, d \rangle = \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)}\, e^{-2\pi i \xi \cdot t}\, dt\,. \tag{6}$$

This transform is known under a dozen names, such as short-time Fourier transform, Gabor transform, radar ambiguity function, coherent state transform, etc., each of which indicates a particular scientific application. Choosing g to be a Gaussian or a smooth compactly supported cut-off function, the coefficient $\langle f, M_\xi T_x g \rangle$ can be taken as a measure for the amplitude of the "frequency" ξ at "time" x.

Using translations and dilations only, the transform becomes

$$\langle f, d \rangle = \langle f, D_a T_x g \rangle = |a|^{-d/2} \int_{\mathbb{R}^d} f(t) g\Big(\frac{t-x}{a}\Big)\, dt\,. \tag{7}$$

This is the (continuous) wavelet transform. By taking g to be compactly supported with $\int_{\mathbb{R}^d} g(t)\, dt = 0$, it describes the local change of f near x in a neighborhood of size a. As a tends to zero, this transform "zooms" in to x and can be used to determine the local smoothness of f at x.

In taking the full time-frequency time-scale dictionary $\mathcal{D}_{\max}$, one obtains the wave packet transform; it combines the virtues of both the short-time Fourier transform and the wavelet transform. For $g(t) = e^{-\pi t \cdot t}$ this transform is also known as the FBI-transform (for Fourier, Bros, Iagolnitzer) and is instrumental for the analysis of wavefront sets in PDE, e.g., [4].

The immediate questions to be asked about a dictionary concern its properties, special construction, and applications. When does a dictionary span $L^2(\mathbb{R}^d)$? Can we construct structured dictionaries that form an orthonormal basis? In particular, do there exist "nice" functions g such that the corresponding Gabor system $\mathcal{D}_G$ or wavelet system $\mathcal{D}_W$ is an orthonormal basis? How do we deal with overcomplete dictionaries? For a given dictionary $\mathcal{D}$, how can a function $f \in L^2(\mathbb{R}^d)$ be represented as a series expansion $f = \sum_{d \in \mathcal{D}} c_d\, d$? How can the coefficients $c_d \in \mathbb{C}$ of such an expansion be determined? How accurately can a function f be approximated by a finite linear combination with respect to the dictionary $\mathcal{D}$?

What are the advantages of special wave packet dictionaries? How are Gabor dictionaries or wavelet dictionaries applied? How can a dictionary be used for the analysis of an operator?

A large part of this book is devoted to answering these questions. The investigation of structured orthonormal bases is particularly interesting. For wavelet dictionaries, a fundamental construction of Ingrid Daubechies [2] exhibits a class of "wavelets" g that are n-times differentiable (for given $n \geq 0$) and compactly supported such that $\mathcal{D}_W = \{D_{2^j} T_k g : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. The construction of such wavelet bases, their parametrization, and many variations are explained in Chapters 1 and 2.

By contrast, there does not exist any "nice" orthonormal basis of time-frequency shifts. The obvious orthonormal basis $\{M_l T_k \chi_{[0,1]} : k, l \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$ lacks smoothness and decay in the frequency domain. The Balian-Low Theorem asserts that this example is typical. If $\mathcal{D}_G = \{M_l T_k g : k, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, then g possesses maximal uncertainty in the sense that $\int\int |g(x)|^2 |\hat{g}(\xi)|^2 x^2 \xi^2 \, dx d\xi = \infty$. Possible remedies are either to investigate modified time-frequency dictionaries of the form $\{\cos(lx)\, g(x - k/2), \sin(lx)\, g(x - k/2) : k, l \in \mathbb{Z}, l \geq 0\}$ (described in Chapter 4) or to abandon the basis property.

Currently there is no systematic theory about mixed dictionaries containing time-frequency shifts and dilations. They first occurred in the theory of Cordoba and Fefferman about pseudodifferential operators [1] and are currently investigated in the context of density theorems and so-called alpha-modulation spaces.

(iii) *Redundancy*: A dictionary $\mathcal{D}$ is called redundant if f is overdetermined by the coefficients $\langle f, d \rangle, d \in \mathcal{D}$. Redundancy is a very useful property in many applications in signal and image processing, because the extra information contained in the redundant coefficients can be used to compensate for the loss or distortion of some of the coefficients $\langle f, d \rangle$. The first question to be answered is how f can be recovered or approximated from the coefficients with respect to $\mathcal{D}$. The best notion to deal with redundant dictionaries is the concept of a *frame*. A countable set $\{d_n : n \in \mathbb{N}\}$ in $L^2(\mathbb{R}^d)$ is called a frame for $L^2(\mathbb{R}^d)$ if there exist positive constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{n=1}^{\infty} |\langle f, d_n \rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)\,. \tag{8}$$

An easy Hilbert space argument yields the existence of a dual frame $\{e_n\}$ such that every $f \in \mathcal{H}$ can be expanded into a series

$$f = \sum_n \langle f, d_n \rangle e_n = \sum_n \langle f, e_n \rangle d_n \, .$$

The frame expansion is similar to an orthonormal expansion, but in general the coefficients are not uniquely determined.

Frames are inevitable in time-frequency analysis, because a Gabor dictionary $\mathcal{D}_G = \{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}^d\}$ with a "nice" $g \in L^2(\mathbb{R}^d)$ cannot be an orthonormal basis. To preserve the simple structure of $\mathcal{D}_G$, one is therefore forced to look at frames of this form [2, 7].

A much more general question concerns the construction and characterization of frames consisting of countable subsets of the maximal time-frequency time-scale dictionary $\mathcal{D}_{\max} = \{M_\xi T_x D_a g\}$. This problem is amazingly rich and offers links to complex analysis, numerical analysis, approximation theory, and sampling theory. Chapter 3 surveys the abstract theory of frames and of Gabor frames, discusses the connection of frames with sampling theory, and presents several numerical reconstruction algorithms.

(iv) *Adaptive decompositions*: In some applications to signal and image processing, the goal is to capture the essential information about a function, a signal, or an image with as few coefficients $\langle f, d \rangle$ as possible. Intuitively, the best approximation of f is obtained by projecting f onto the subspace spanned by those $d_n \in \mathcal{D}$ that correspond to the largest coefficients $|\langle f, d_n \rangle| \geq |\langle f, d \rangle|$ for $d \neq d_1, \dots d_n$. Since the optimal subspace to which f is projected depends on f, the approximation is called *adaptive*. When only few coefficients are required for a good approximation of f, then f is *sparse* with respect to the given dictionary. As a general principle, sparsity with respect to a time-frequency or time-scale dictionary is related to the smoothness of f. For instance, if a function is sparse with respect to a wavelet dictionary, then it belongs to a Sobolev space or to a Besov space.

Similarly, for an operator A on a Hilbert space $\mathcal{H}$ one may consider the matrix entries $\langle Ad, d' \rangle, d, d' \in \mathcal{D}$, with respect to a dictionary. By keeping only the large entries, one may hope to obtain a good approximation of A. The art is now to design a dictionary in which only few entries are large. Then A possesses a sparse approximation with respect to $\mathcal{D}$. A large class of pseudodifferential operators is sparse with respect to Gabor dictionaries, whereas singular integral operators, precisely the class of Calderòn-Zygmund operators, are sparse with respect to wavelet dictionaries. These ideas are highlighted in Chapter 6. The analysis of operators with wave packet dictionaries is more subtle; some sketches are given in Chapter 7.

This book. *Time-Frequency and Time-Scale Methods* by J. Hogan and J. Lakey covers an enormous range in time-frequency analysis and time-scale analysis. The topics range from sampling algorithms for trigonometric polynomials to the Fefferman-Phong eigenvalue estimates for Schrödinger operators, from the construction of orthonormal wavelet bases to the investigation of functions of bounded variation with wavelets, from the standard uncertainty principle to the discussion of the bilinear Hilbert transform with wave packets. The common thread of all these topics is the use of time-frequency or time-scale dictionaries and the aspects of dictionaries, frames, and adaptive decompositions.

This book is not an introduction. To read it with profit, the reader should know the basic material on wavelets, as contained in the first chapters of [2, 11], and of time-frequency analysis, as outlined in [4, 7].

This book is about methods and ideas. The main goal is to motivate, discuss, and describe important results and new developments in time-frequency and time-scale analysis. Full proofs are rarely given, but insightful sketches of proofs or at least descriptions of the underlying ideas are given for most results. This style makes it possible to cover an enormous amount of material, and indeed the book reviews the main results of (an estimated) 200 articles in some detail. The organization of the book is therefore non-standard: the order of presentation is dictated by relationship of topics, but not by the hierarchy of prerequisites and the logic of proofs. Each chapter, if written in a self-contained style with full proofs and all technical details, would easily fill an entire book by itself. The book can therefore not be compared to any of the existing textbooks on harmonic analysis, wavelet theory, or time-frequency analysis.

The advantage of this presentation is an organic and coherent picture of a huge body of literature. The reader is spared most technical details and can focus on the inner working of ideas. The disadvantage is that it is not easy to learn from this book directly. It is probably best used as a first orientation and guide about methods and ideas. For an in-depth study of the discussed topics, the reader should then consult either the original references or one of the available textbooks.

The resulting book is like a gigantic mosaic. Each facet describes an interesting result. While neighboring facets are not necessarily related in any formal way, the entire mosaic, when viewed from a distance, displays a coherent and magnificent picture, the picture of time-frequency and time-scale analysis. The book is a great resource for experts to obtain an overview of the field and its literature. It is also recommended for advanced graduate students who want to escape their specialization and expand their horizons to a broader view of their field.

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