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J-holomorphic curves and symplectic topology, by Dusa McDuff and Dietmar Salamon, Colloquium Publications, vol. 52, Amer. Math. Soc., Providence, RI, 2004, xii+669 pp., US$99.00, ISBN 0-8218-3485-1

In his 1985 paper [12] M. Gromov introduced striking new techniques into symplectic geometry: J-holomorphic, or pseudo-holomorphic curves. Given two almost complex manifolds (X, J_X) and (Y, J_Y) of real dimension $2k$ and $2n$, a map $f : X \to Y$ is called holomorphic (or pseudo-holomorphic, or (J_1, J_2)-holomorphic) if its differential df is a complex linear map $(TX, J_X) \to (TY, J_Y)$, i.e.

$$J_Y \circ df = df \circ J_X \,. \tag{1}$$

A problem with this definition is that when $k > 1$, the system (1) is overdetermined, unless both almost complex structures are integrable. Thus generically the system (1) has no solutions, even locally. However, when $n = 1$, i.e. when (X, J_X) is a Riemann surface, then the system is determined. In this case $j = J_X$ is always integrable; i.e. (X, j) is a Riemann surface, and locally there is the same degree of freedom of constructing (j, J)-holomorphic maps whether $J = J_Y$ is integrable or not. In this case J-holomorphic maps $(X, j) \to (Y, J)$ are usually called *J-holomorphic curves*.

If X is closed, or with appropriate boundary conditions, the system (1) is elliptic, and its linearization has the same principal symbol as the usual $\bar\partial$-operator. Hence, one can develop the Fredholm theory for J-holomorphic curves and study their moduli spaces, which under certain transversality conditions are finite-dimensional orbifolds.

The Gromov compactness theorem says that if the target manifold Y is compact or satisfies some geometric bounds at infinity, then the moduli spaces of holomorphic curves of bounded area can be compactified by adding *stable nodal curves*, in the spirit of Deligne-Mumford compactification of the moduli space of Riemann surfaces.

Gromov also observed that one can ensure automatic area bounds if Y admits a symplectic form ω which *tames* J. The taming condition means that the symplectic form ω is positive on tangent complex lines; i.e. $\omega(v, Jv) > 0$ for all non-zero tangent vectors $v \in TX$. Indeed, in this case one can bound the area $Area(f)$ of a holomorphic curve $f : X \to Y$ in terms of its symplectic area:

$$\mathrm{Area}(f) \le C \int\limits_X f^*\omega,$$

where the constant C depends only on the geometry of Y. On the other hand, when X is closed the integral $\int_X f^*\omega$ depends only on the homology class of the curve, which guarantees a bound on the area of holomorphic curves in a given homology class. Thus the symplectic form ω literally tames J, i.e. does not allow J-holomorphic curves to misbehave!

The compactness property of moduli spaces of holomorphic curves implies that some homological characteristics of the compactified moduli spaces depend not

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on J but only on its taming symplectic form. This observation led Gromov to the first genuinely symplectic invariants, both for symplectic domains and closed symplectic manifolds. To clarify the meaning of the term "genuinely symplectic", let us observe that given a symplectic form ω on a closed $2n$-dimensional manifold M there are two obvious quantities which are invariant under the action of the identity component of the group of symplectomorphisms. These are the cohomology class $[\omega] \in H^2(M)$ and the homotopy class of ω in the space of all non-degenerate but not necessarily closed 2-forms (this space is homotopy equivalent to the space of almost complex structures on M). In a similar vein, for an open symplectic manifold (U, ω), say a domain in the standard symplectic space $(\mathbb{R}^{2n}, \omega = \sum_1^n dp_i \wedge dq_i)$, the symplectic volume $\int_U \omega^n$ is its obvious symplectic invariant. The question which was open before Gromov's work is whether there are any other, not cohomological or bundle-theoretic, symplectic invariants. Here is a sketch of the definition of the famous *Gromov width* for the case of a domain in the standard symplectic space $\mathbb{R}^{2n}$. Suppose we are given a domain $U \subset \mathbb{R}^{2n}$ with compact closure. Let us choose a point $p \in U$ and an almost complex structure J on $\mathbb{R}^{2n}$ tamed by ω. Let $f : D^2 \to \mathbb{R}^{2n}$ be a J-holomorphic map of the unit disc such that $f(0) = p$ and $f(\partial D^2) \subset \mathbb{R}^{2n} \setminus U$. Next define

$$A_{p,J,f}(U) = \int\limits_{f^{-1}(U)} f^*\omega,$$

and then set

$$\mathrm{width}(U) = \sup_J \inf_f A_{p,J,f}(U).$$

The width(U) is independent of a choice of the basepoint p because the group of symplectomorphisms acts transitively on points. The Gromov width is obviously monotone under embeddings of domains and remains unchanged if one re-embeds symplectically the closure of the domain U. Gromov computed the width of balls $B^{2n}(R) = \{\sum_1^n p_i^2 + q_i^2 < R^2\}$ and polydiscs $P(a_1, \dots, a_n) = \{p_j^2 + q_j^2 < a_j^2, j = 1, \dots, n\}$:

$$\mathrm{width}(B^{2n}(R)) = \pi R^2 \quad \text{and} \quad \mathrm{width}(P(a_1, \dots, a_n)) = \pi \min_j (a_j^2),$$

which implied his famous non-squeezing theorem: one cannot symplectically embed a unit ball into a narrow polydisc, even if the latter has large volume.

A certain twist of the theory of J-holomorphic curves led to *Floer homology theory* [7, 8] and *Hofer geometry* [13] with deep applications in Hamiltonian Dynamics (see, for instance, [19]). Gromov's theory of J-holomorphic curves came just in time to provide foundations for some exciting new developments in Mathematical Physics, and especially String Theory. This led to the first mathematical proof [10] of the Mirror Symmetry phenomenon observed by physicists and also created a large and active area of *Gromov-Witten theory* which employs along with symplectic geometers also specialists in many other areas: notably algebraic geometers, specialists in integrable systems, random matrix theory, representations, etc.

In his seminal paper [12] Gromov already observed that J-holomorphic curves have especially remarkable applications in 4-dimensional symplectic geometry. For instance, he proved that an open 4-manifold M which at infinity is the standard symplectic 4-space has to be symplectomorphic to $\mathbb{R}^4$ via a symplectomorphism fixed at infinity, provided that M satisfies some constraints on its second homology group. Dusa McDuff [16] later strengthened Gromov's result by showing that any such M is the standard symplectic $\mathbb{R}^4$ up to symplectic blow-ups. Taubes's theorem "Gromov-Witten = Seiberg-Witten" (see [20]) paved the way for further deep applications of J-holomorphic curves, in 4-dimensional topology. Using J-holomorphic curves, Ozsváth-Szabó theory [18] yielded spectacular applications in 3-dimensional topology.

Today the theory of J-holomorphic curves has become a necessary part of education, not only of a large army of mathematicians working in Gromov-Witten theory, symplectic geometry and Hamiltonian Dynamics, but also of specialists in many other areas of Mathematics: low-dimensional topology, Mathematical Physics and, in particular, the mathematical theory of Mirror Symmetry, the theory of integrable systems, random matrix theory, combinatorics, etc.

However, the textbook literature on the subject of J-holomorphic curves is very limited, and the book under review has taken an important step towards filling the gap. As the authors write in their preface, the book started its life as a second edition of the book *J-holomorphic Curves and Quantum Cohomology* [17], written by the same authors in 1996. However, even a brief look at the covers of the two books immediately reveals that the book under review should be considered much more than just a second edition: it is more than 3 times thicker!

Even those chapters of the book which follow the skeleton of the first book are completely rewritten and expanded. Several new chapters, and especially many important appendices, are added (there are more than 100 pages of them). However, despite its size this book does not even attempt to be an encyclopedia of Gromov-Witten theory. As was pointed out above, nowadays Gromov-Witten theory is a very large area which is of interest to a very broad spectrum of readers. All of them are interested in different aspects of the theory, and it seems to be an almost impossible task to write a book which would satisfy all potential readers.

One of the main goals of the book is to set up the rigorous foundations of the theory. This was, and to a large extent still remains, a serious problem in the area. The development went so fast that many authors were not careful enough in setting up the proper analytic foundations. The book under review does not attempt to do it in a maximal generality. In particular, the book does not even touch the theory of holomorphic curves of genus > 1. Also in most cases the authors restrict the discussion to the so-called semi-positive case, when there is a straightforward way to handle the transversality issues, and thus avoid the virtual cycle theory. But what the authors discuss in the book is reliable and can be safely used for applications. I would say that it sets a certain standard for how these foundational matters should be handled.

Let us briefly review the content of McDuff-Salamon's book. The first two chapters are devoted to local and global analytic setup. The authors begin with the local theory of J-holomorphic curves and also discuss the positivity of the intersection property for holomorphic curves in symplectic 4-manifolds, which is the basis for most applications of J-holomorphic curves in low-dimensional topology. Let me note here that related matters are also discussed in great detail in one of the appendices to the book. Next, they move to the global functional analytic setup, discuss the transversality issue, which is one of the subtle points of the theory, and applications of the implicit function theorem in the situation when transversality is achieved. One of the advantages of Gromov's method of J-holomorphic curves is that it allows us to work in the flexible world of almost complex structures rather than in a rigid algebro-geometric world. In particular, it is much easier to achieve transversality here. On the other hand, algebraic-geometric objects, even in the absence of transversality, are never too bad. In particular, one has in Algebraic Geometry a well developed technique of obstruction bundles to deal with the non-transversality issues. However, for a general almost complex structure J, in the absence of transversality, the moduli spaces of J-holomorphic curves can be arbitrary closed sets in appropriate functional analytic spaces. Hence, it is crucial to be able to achieve transversality when working with J-holomorphic curves. Unfortunately, this is not always possible in the class of J-holomorphic curves, and one needs to work with more general objects. To avoid complications, McDuff and Salamon restrict themselves in their book to the situation when this is possible.

The next two chapters, IV and V, are devoted to the Gromov compactness theorem for J-holomorphic curves. In particular, the bubbling phenomenon and the notion of a stable nodal curve are discussed here. Chapters VI and VII contain the definition and property of genus 0 Gromov-Witten invariants, while the analytic gluing technique, necessary for proving some algebraic identities for GW-invariants, is postponed to a later discussion in Chapter X. Chapter VII culminates with a sketch of the proof of Kontsevich's recursion formula (see [14]) for the numbers N_d of rational holomorphic curves in $\mathbb{C}P^2$ of degree d passing thorough $3d-1$ points. In Chapter VIII the authors develop the theory of holomorphic sections of symplectic fibrations. This is an important subject tightly related to the geometry of symplectic transformations and Hamiltonian Dynamics.

The main symplectic geometric applications of the theory of J-holomorphic curves are collected in Chapter IX. This is a huge subject, and of course, any comprehensive survey of the results in this area would have to go far beyond the scope of this book. Instead, the authors tastefully choose examples in the main directions of such applications.

The development of symplectic topology was greatly influenced by the Arnold conjecture [1] about the lower bound for the number of fixed points of symplectic transformations and the Weinstein conjecture [21] about existence of periodic orbits of Reeb fields. Note that despite remarkable progress, neither of these conjectures is yet established in full generality. Any serious discussion of the results related to the Arnold conjecture requires Floer homology theory, which is discussed in Chapter XII of the book. Thus in Chapter IX the authors prove just a qualitative statement towards the Arnold conjecture, i.e., that there exists at least one fixed point of a Hamiltonian diffeomorphism, subject to some additional conditions on the symplectic manifold. In this form (for a slightly smaller class of manifolds) the theorem was first proven in Gromov's original paper [12] on pseudo-holomorphic curves.

Another classical area of applications of J-holomorphic curves is the Lagrangian intersection theory and the topology of Lagrangian submanifolds. The book contains, in particular, the proof of Gromov's theorem about the existence of a non-constant holomorphic disc with boundary in a Lagrangian submanifold of $\mathbb{C}^n$, which implies that an embedded Lagrangian submanifold of $\mathbb{C}^n$ cannot be exact; another of Gromov's theorems which states that any exact Lagrangian submanifold of a cotangent bundle should intersect the 0-section; Lalonde-Sikorav's theorem [15] that an exact Lagrangian submanifold of a cotangent bundle projects onto the 0-section; and several other results which have already become classic.

Other symplectic applications considered in this chapter include further results from Gromov's seminal paper, together with their later development: non-squeezing theorems, special properties of symplectic 4-manifolds, topology of the group of symplectomorphims of certain 4-manifolds. There is also a discussion of the Hofer geometry of the group of Hamiltonian diffeomorphisms, as well as of the problem of distinguishing symplectic manifolds via their Gromov-Witten invariants.

Chapter XI is devoted to a systematic study of properties of genus 0 Gromov-Witten invariants. The genus 0 Gromov-Witten invariants of a manifold X allows one to deform the cup-product on $H^*(X)$ into the *quantum* cup product with the coefficients in the so-called Novikov ring. In turn, the quantum multiplication leads to a structure of a Frobenius manifold, as was discovered by B. Dubrovin in [4]. Besides the general theory, the chapter contains several interesting examples of computations: Fano toric manifolds, flag manifolds and Grassmanians. There is also a discussion of the Mirror Symmetry conjecture for Calabi-Yau manifolds, and in particular of the case of the quintic in $\mathbb{C}P^5$, which was solved by A. Givental, [10]. A special section in this chapter is devoted to properties and applications of Seidel representation, which associates with a Hamiltonian symplectomorphism an element in quantum cohomology of the manifold.

The final chapter, Chapter XII, is a sketch of Floer homology theory. Floer homology theory began with an observation which goes back to Conley and Zehnder [3] that the gradient trajectories for the action functional on the loop space of a symplectic manifold M can be interpreted as J-holomorphic curves for a suitable almost complex structure (not necessarily on M itself). In combination with Gromov's theory of J-holomorphic curves this led Floer to the creation of the symplectic Floer homology theory. There are essentially two different, though tightly related, symplectic Floer homology theories, which were originally designed to deal with two Arnold conjectures: about fixed points of symplectomorphisms and about intersections of Lagrangian manifolds. The Lagrangian Floer homology theory is only mentioned in the book, while the former theory is systematically studied for the semi-positive case. Symplectic Floer homology theories are usually filtrated by the value of the action functional. This leads to the theory of spectral invariants. The Gromov width, Ekeland-Hofer capacities and the Hofer norm are among examples of such invariants. This chapter also contains a lot of interesting and non-conventional material, such as a discussion of the notion of Donaldson quantum category, as well as the symplectic vortex equation, whose theory was developed by D. Salamon and his collaborators.

As was mentioned above, the book contains over 100 pages of very essential appendices. There are five of them. The first three contain the standard necessary analytic material. Their inclusion in the book is very convenient for the reader. The last two appendices describe in detail Grothendieck-Knudsen compactification of the moduli space $\mathcal{M}_{0,n}$ of configuration of n marked points in $\mathbb{C}P^1$ and local properties of J-holomorphic curves near their singularities. The latter theory is especially crucial for the 4-dimensional applications of J-holomorphic curves.

D. McDuff and D. Salamon have written a very good and useful book. It is already widely used as a standard graduate textbook for students studying Gromov-Witten theory and, more generally, for those interested in Symplectic Geometry, Hamiltonian Dynamics, Low-Dimensional Topology and several other related areas. It is also an excellent reference book for specialists.

Though Symplectic Topology is not yet even 30 years old, it has already witnessed spectacular achievements and progress. Its methods have penetrated many different areas of Mathematics, while, in its turn, it heavily borrows ideas from other areas of Mathematics and Physics. In recent years further progress has been achieved in understanding the structure and computation of higher genus Gromov invariants (e.g. [11]), as well as the structure of relative invariants (based on J-holomorphic curves with Lagrangian boundary conditions); see [9]. The contact homology theory [5, 2] allowed us to define new invariants of contact manifolds and their Legendrian submanifolds. A more general project of Symplectic Field Theory [6] attempts to provide a unified approach to Gromov-Witten theory, Floer homology and contact homology theories.

On the other hand, despite remarkable achievements, many basic questions remain unanswered. While J-holomorphic curves continue to be a source of new discoveries and important and interesting applications, they have proven themselves to be a very efficient tool in 4-dimensional symplectic topology, while higher-dimensional applications so far have been more modest. It seems that the time is ripe for creation of the next generation of analytic tools.

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