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Non-linear elliptic equations in conformal geometry, by S.-Y. Alice Chang, Zürich Lectures in Advanced Mathematics, European Mathematical Society, Zürich, 2004, viii+92 pp., US$28.00, ISBN 3-03719-006-X

A conformal transformation is a diffeomorphism which preserves angles; the differential at each point is the composition of a rotation and a dilation. In its original sense, conformal geometry is the study of those geometric properties preserved under transformations of this type. This subject is deeply intertwined with complex analysis for the simple reason that any holomorphic function $f(z)$ of one complex variable is conformal (away from points where $f'(z) = 0$), and conversely, any conformal transformation from one neighbourhood of the plane to another is either holomorphic or antiholomorphic. This provides a rich supply of local conformal transformations in two dimensions. More globally, the (orientation preserving) conformal diffeomorphisms of the Riemann sphere S^2 are its holomorphic automorphisms, and these in turn constitute the non-compact group of linear fractional transformations. By contrast, the group of conformal diffeomorphisms of any other compact Riemann surface is always compact (and finite when the genus is greater than 1). Implicit here is the notion that a Riemann surface is a smooth two-dimensional surface together with a conformal structure, i.e. a fixed way to measure angles on each tangent space. There is a nice finite dimensional structure on the set of all inequivalent conformal structures on a fixed compact surface; this is the starting point of Teichmüller theory. All of this is in accord with the fact that the conformal property is an elliptic equation in two dimensions, so there are many solutions locally, but global existence is constrained and solutions lie in finite dimensional families.

In higher dimensions, the equation that a map be conformal is formally overdetermined, so one expects very few such transformations, if any. Generalizations of the linear fractional transformations always do exist: for any n, the conformal group $O(n+1,1)$ – also called the Möbius group – consists of all mappings on $\mathbb{R}^n$ which are compositions of translations, rotations, dilations and inversions about spheres. Any Möbius transformation extends naturally both to a global conformal diffeomorphism of S^n, and also to its interior, the ball, as an isometry for the hyperbolic Poincaré metric on B^{n+1}. This last statement is the basis for the far-reaching correspondence between hyperbolic manifolds and conformal group actions on the sphere, which has many ramifications in geometry, dynamics and number theory. However, these are the *only* conformal mappings of the sphere; a theorem due to Liouville states that any locally defined conformal diffeomorphism in $\mathbb{R}^n$ with $n > 2$ is the restriction of one of these Möbius transformations. (In particular, except for very special cases, holomorphic maps of several complex variables are never conformal.)

Ideas from conformal geometry pervade 2 dimensional geometry and topology, but as just indicated, in order to develop a satisfactory theory with a broad scope in higher dimensions, it is a mistake to place the emphasis on mappings as the central objects of interest. Rather, one should change focus from mappings to

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conformal structures and the notion of conformal equivalence of metrics. Two Riemannian metrics g and g' on an n-dimensional manifold M, $n \geq 2$, are called conformally equivalent if $g' = e^{2\phi} g$ for some (smooth) function ϕ. The conformal class $[g]$ consists of all metrics related in this way to a given metric g. There is a significant difference between the 2- and the higher dimensional settings since the space of inequivalent conformal structures on a closed n-manifold, $n \geq 3$, is always infinite dimensional. Conformal geometry, as it is now understood in geometric analysis, involves the study of conformal classes and various objects associated to them, e.g. conformally covariant differential operators and conformal invariants associated to them, conformal anomaly formulæfor global spectral invariants, and canonical metrics within a conformal class, to name just a few. I cannot do justice here to even just these topics, let alone other parts of the subject which touch on Lie and representation theory, as well as other parts of PDE and geometry. What follows is a brief overview of an intriguing web of ideas which has been the crux of a large body of research in the last decade and which is the subject of the book under review.

I begin with the Laplacian on a surface M, its functional determinant, and the relationship of these with Gauss curvature. To any Riemannian metric g on an n-dimensional manifold, one can naturally associate a second order elliptic differential operator acting on functions; this is the Laplacian, usually denoted Δ_g. In two dimensions, this operator is conformally covariant in the sense that if $g' = e^{2\phi} g$, then $\Delta_{g'} = e^{-2\phi} \Delta_g$. Very closely related is the formula giving the relationship between the Gauss curvatures, K_g and $K_{g'}$, of these two conformally related metrics:

$$\Delta_g \phi + K_{g'} e^{2\phi} = K_g.$$

The area elements of these two metrics are related by $dA_{g'} = e^{2\phi} dA_g$ and by the divergence theorem, $\int_M \Delta_g \phi \, dA_g = 0$, so we deduce from this formula that

$$\int_M K_{g'} \, dA_{g'} = \int_M K_g \, dA_g;$$

in other words, this total curvature integral is independent of the choice of metric in a conformal class. Of course, more is true: the Gauss-Bonnet theorem states that the integral of the Gauss curvature over a closed surface is equal to 2π times the Euler characteristic $\chi(M)$ and hence is also independent of the conformal class.

One old problem, studied by Nirenberg, Kazdan-Warner and many others, asks whether given an arbitrary smooth function on a closed surface, it is possible to find a metric in a fixed conformal class with Gauss curvature equal to this prescribed function. The special case where this function is constant is equivalent to the uniformization theorem. However, if the function is non-constant, this problem is in some sense unstable and is known to be rather difficult to solve, though there are many partial results.

Next, let A be a self-adjoint elliptic operator of order d on a compact n-dimensional manifold. Its spectrum consists of the sequence of eigenvalues $\{\lambda_j\}$, so for each λ_j there exists a function ϕ_j with $A\phi_j = \lambda_j \phi_j$. We assume that every $\lambda_j \geq 0$ (so we shall take A to be $-\Delta_g$ momentarily). To understand this eigenvalue sequence it is convenient to wrap these numbers up into a single function depending on an auxiliary variable. A good choice is the zeta function of A, $\zeta_A(s) := \sum_{\lambda_j > 0} \lambda_j^{-s}$. By the Weyl asymptotic theorem one knows that $\lambda_j \sim \alpha j^{d/n}$ for some $\alpha > 0$ which is independent of j. This implies in turn that $\zeta_A(s)$ is holomorphic in the right half-plane $\mathrm{Re}(s) > n/d$. Using the short time asymptotics of the heat equation associated to A, which is closely connected to the local geometry, it can be shown that $\zeta_A(s)$ continues meromorphically to the entire complex plane and is regular at $s = 0$. The definition proposed by Ray and Singer in the early 1970's is that $-\log\det A = \zeta'(0)$, which is equivalent to the usual answer when A is a positive symmetric matrix of finite rank (try it!). This determinant is a spectral invariant, i.e. a numerical invariant of the operator A which depends on the entire spectrum (and nothing else). In particular, $g \mapsto F(g) := -\log\det(-\Delta_g)$ is a functional on the space of all metrics.

A remarkable fact, discovered by the physicist Polyakov, is that despite the global nature of the spectrum, if $A = -\Delta_g$ on a compact surface, then the differential of this functional F in conformal directions (i.e. only allowing variations of the metric within a conformal class) is given by a *local* formula: if ϕ_ϵ is any family of functions depending smoothly on a parameter ϵ with $\phi_0 = 1$, $\dot{\phi}_0 = w$, and if $g_\epsilon = e^{2\phi(\epsilon)}g$, then

$$DF_*|_g\,(w) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(g_\epsilon) = \frac{1}{12\pi}\int_M (|\nabla w|^2 + 2K_g w)\, dA_g.$$

This expression is called 'local' because it is an integral of quantities which are all given in terms of derivatives of w and the local geometry of g.

It is no surprise that the critical points of F within a conformal class might be of interest. To eliminate the trivial scaling dependence, we restrict ourselves to metrics with fixed area, i.e. $\int_M e^{2\phi_\epsilon}\, dA_g = \text{const.}$; differentiating this contraint with respect to ϵ shows that $\int_M w\, dA_g = 0$. It is then a simple exercise in the calculus of variations to see that g is a critical metric if and only if K_g is constant. This is a new characterization of the constant curvature metric(s) in a conformal class as critical points of the 'log det' functional. In an influential series of papers in the late 1980's, Osgood, Phillips and Sarnak made these observations and went on to prove the compactness of the set of metrics g on a closed surface M, not necessarily all in a fixed conformal class, for which the spectrum of Δ_g is a fixed sequence $\{\lambda_j\}$. (The crucial extra ingredient is that F is a proper function on the space of all constant curvature metrics of fixed area.)

The significant features of this two dimensional story are the conformal covariance of the Laplacian in two dimensions, its conformal transformation rule and the relationship between the Gauss curvatures of two conformally related metrics; all of these in turn are related to the conformal variation of $\det(-\Delta_g)$.

Determinants of Laplacians in higher dimensions are less easy to study, and in particular, on a compact manifold (M, g) with $\dim M > 2$ there is no local formula for the derivative, even just in conformal directions, of $F(g) = -\log\det(-\Delta_g)$. Okikiolu has some nice results concerning the variations of this functional in all (not just conformal) directions near certain natural metrics (such as round spheres) in arbitrary dimension. However, the most promising avenue of research requires conformal covariance of the operator. This sets the program: find geometrically natural, elliptic, conformally covariant operators in any dimension; understand 'curvatures' associated to these operators and any other curvature quantities which transform simply under conformal changes, as well as the total integrals of these curvatures; and study the determinants of these putative conformally covariant operators and their Polyakov formulæ.

One conformally covariant operator has been known for a long time. This is the conformal Laplacian,

$$L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g.$$

Here R_g is the scalar curvature function on M (whose value at a point p is the average of all the sectional curvatures amongst all 2-planes in the tangent space at that point). If $g' = e^{2\phi} g$ as usual, then for any function w,

$$L_{g'} w = e^{-\frac{n+2}{2}\phi} L_g(e^{\frac{n-2}{2}\phi} w).$$

Another form of this covariance is obtained by setting $u^{4/(n-2)} = e^{2\phi}$, which gives $L_{g'} w = u^{-(n+2)/(n-2)} L_g(u\, w)$. In particular, applying both sides to the constant function $w = 1$ yields

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n-2}{4(n-1)} R_{g'} u^{\frac{n+2}{n-2}} = 0.$$

This is the famous Yamabe equation, which relates the scalar curvature functions of any two conformally related metrics. Note that it does not make sense to set $n = 2$ here, but there is still a way to think of the transformation rule for Gauss curvature in two dimensions as a limiting case "as $n \to 2$" of this formula. There is a long history concerning the existence of solutions of this semilinear elliptic equation when $R_{g'}$ is a prescribed function and in particular when $R_{g'}$ is constant. As in the two dimensional case, this has a particularly delicate type of non-linearity which is borderline with respect to 'standard techniques', and the eventual solution of this problem (for constant $R_{g'}$) by Schoen, following earlier work of Yamabe, Trudinger and Aubin, is well documented in the marvelous detailed survey [1]. The ideas surrounding this, as well as the case where $R_{g'}$ is not constant, have provided major stimulus in geometric analysis and PDE since the 1960's, and this earlier story was the true beginning of sophisticated analytic techniques in conformal geometry. Of particular note is the work of Schoen and Yau concerning the structure of conformally flat manifolds with positive scalar curvature. Here, conformal flatness is the condition that the metric be locally conformally equivalent to the Euclidean one. There is a lot of interesting, and still not completely understood, geometry behind this condition, and it has also been an important preliminary hypothesis in many of the analytic investigations in this area. The monograph [2] provides a good treatment of many topics relevant to this. Something rather surprising, which emerged first in the work of Schoen and more recently in that of C.S. Lin, Hebey and his students, etc., is that this specific geometric equation has many special properties not shared by even very small 'non-geometric' perturbations of the equation.

Prompted by questions in quantum field theory, in the early 1980's the young mathematical physicist Steve Paneitz discovered a fourth order conformally covariant operator in any dimension $n \geq 4$. When $n = 4$, this 'Paneitz operator' is

$$P_g = \Delta_g^2 + \delta(\frac{2}{3} R_g\, g - 2\mathrm{Ric}(g))d.$$

(Here d maps functions to 1-forms, δ is its adjoint and maps 1-forms to functions, and $(2/3)R_g\, g - 2\mathrm{Ric}(g)$ acts as a linear transformation on 1-forms.) There are corresponding, but somewhat more complicated, expressions in higher dimensions. It satisfies, with $g' = e^{2\phi} g$ and for any w,

$$P_{g'} w = e^{-\frac{n+4}{2}\phi} P_g(e^{\frac{n-4}{2}\phi} w).$$

In particular, when $n = 4$, $P_{g'} = e^{-4\phi} P_g$. Moreover, there is a new curvature associated with P_g. This has been christened the 'Q-curvature'; when $n = 4$,

$$Q_g = \frac{1}{12}\left(R_g^2 - 3|\mathrm{Ric}(g)|_g^2 - \Delta_g R_g\right) \qquad \text{and} \qquad P_g\phi + 2Q_g = 2Q_{g'}e^{4\phi},$$

and one can still deduce that $\int_M Q_g \, dV_g$ is independent of the choice of metric in a conformal class. It is also interesting to note that this is a summand in the Chern-Gauss-Bonnet formula in this dimension. When $n > 4$, Q_g is (up to a constant) just the term of order zero in the corresponding Paneitz operator. Notice that all of this is very much as for the Laplacian and Gauss curvature in two dimensions (although this Q curvature integral is just a conformal invariant and no longer independent of conformal class); when $n > 4$, these objects are analogous to the conformal Laplacian and scalar curvature. Some part of this was discovered, and certainly all of it was formulated, in a particularly useful way at roughly the same time by Tom Branson.

Paneitz died quite tragically not long after this discovery, but this operator pointed the way toward the general phenomenon. It is now known, through the work of Graham, Jenne, Mason and Sparling, that for n odd and every $k \in \mathbb{N}$, there exists a conformally covariant operator P_n^k of order $2k$, naturally associated to any metric g, which is a lower order perturbation of Δ_g^k; when n is even, this operator exists only when $1 \leq k \leq n/2$. A recent result by Graham and Zworski reconstructs these and obtains some of their properties through an elegant connection with scattering theory on asymptotically hyperbolic manifolds. Associated to each P_n^k is a Q curvature, Q_n^k, which is again the term of order 0 in P_n^k unless n is even and $k = n/2$. It is a higher order analogue of the Gauss or scalar curvature and satisfies a nice transformation rule involving P_n^k.

This is an appropriate place to mention a large program on parabolic invariant theory initiated by Charles Fefferman and its relationship to some of the results above. There is a famous result by Weyl concerning the classification of scalar invariants of Riemannian metrics; these are quantities defined in terms of the metric and its derivatives in any coordinate system, but which are independent of choice of coordinates. This is related to representations of the orthogonal group. Weyl showed that these invariants can all be obtained as contractions of products of the Riemann curvature tensor and its covariant derivatives. Fefferman's goal is to understand scalar invariants of a conformal class (this is 'the real case') or a CR structure (the complex case) or the geometric structures associated to other parabolic groups. Fefferman and Graham laid out a compelling structure and plan of action for accomplishing this, and in recent years this has borne fruit, particularly in the real and complex cases due to the concerted effort of these two and their collaborators. In particular, the original proof of existence of the conformally covariant operators P_n^k was motivated by and obtained as part of this program, while the newer perspective on these operators stems from their basic idea of associating to a conformal structure on a manifold M a canonical complete asymptotically hyperbolic (Riemannian) Einstein metric on a manifold X with $\partial X = M$, so that Riemannian invariants for this filling metric correspond to conformal invariants on the boundary. A new approach to all of these ideas, the tractor calculus, has recently been developed by Rod Gover.

To return to the main thread, Branson and Ørsted proved that there is a Polyakov-type formula for *any* conformally covariant (semibounded) elliptic operator. In 4 dimensions, for operators of order 2, the integrand in this formula is always a linear combination of three basic components: the norm squared of the Weyl curvature tensor, $|W_g|_g^2$, the Laplacian of the scalar curvature, $\Delta_g R_g$, and the Q curvature! Unlike the two dimensional case, however, the existence of critical points for this functional, let alone a characterization of them, is very difficult. This existence was accomplished by Chang and Yang in 4 dimensions, assuming a certain condition on the signs of the constants which arise in the linear combination above (satisfied for the conformal Laplacian) as well as a bound on the integral of the Q curvature (which was later shown by Gursky to be automatically satisfied if one is working in a conformal class with positive Yamabe invariant). However, the existence theory for such extremal metrics remains mostly open, and the geometric nature of these extremal metrics is not clear.

These considerations provide some motivation for the problem of finding metrics within a conformal class with constant Q curvature. Before describing this, I need to introduce one (final!) family of curvature quantities with good conformal transformation properties. These are based on the Schouten tensor $A_g = (n-2)^{-1}(\mathrm{Ric}(g) - (R_g/2(n-1))g)$, which satisfies the transformation rule that for $g' = e^{2\phi}g$,

$$A_{g'} = -\nabla\nabla\phi + d\phi \otimes d\phi - \frac{1}{2}|\nabla\phi|_g^2 + A_g.$$

The scalar invariants which can be derived from this are the k^{th} symmetric polynomials of its eigenvalues, denoted $\sigma_k(A_g)$, which I will call simply the σ_k curvatures. This general class of functionals was discovered by Viaclovsky. There are several connections with the preceding discussion. First, observe that $\sigma_1(A_g)$ is a multiple of the scalar curvature. Next, when n is even, $\sigma_{n/2}(A_g)$ is closely related to $Q_n^{n/2}$; $\int_M \sigma_{n/2}(A_g)\, dV_g$ is an invariant of the conformal class both when $n = 4$ and also in higher dimensions under certain assumptions on the conformal class. This leads to the problem of finding metrics in a given conformal class for which $\sigma_k(A_g)$ is constant (or equal to some prescribed function). The case $k = 1$ is the Yamabe problem, which was discussed above. When $k > 1$, however, this problem becomes a fully non-linear equation. As is the nature of such equations, it is elliptic (and hence admits a good existence and regularity theory) only under special circumstances; in other words, if one starts with a background metric g, then this σ_k function is elliptic for ϕ near 0 only if g satisfies some potentially restrictive conditions. In fact, when $k \geq n/2$, it follows by a purely algebraic calculation that if the σ_k function is elliptic at some metric g', then the Ricci tensor of g' is positive. This is quite stringent, and for k in this range, the existence of metrics with $\sigma_k > 0$ gives strong topological restrictions. At any rate, this is quite different from the situation when $k = 1$, where the problem can always be treated by elliptic methods.

In the last few years, much progress has been made, and there is now a very substantial literature on the σ_k equations and the existence of metrics in a conformal class for which the σ_k curvature is constant. Interestingly, this equation is not always the Euler-Lagrange equation for an 'energy', and this problem may not be approached by methods in the calculus of variations, except in certain special cases. For example, the equation does have a variational structure when the conformal class is locally conformally flat; using this and other ideas, this case has been treated quite thoroughly by A. Li and Y. Li and by Guan and G. Wang. In the non-locally conformally flat case, Gursky and Viaclovsky proved existence when $k > n/2$ if one has a good initial metric (for which the problem is elliptic), and there are more recent results by Trudinger and X. Wang. The 4-dimensional case was handled by Chang-Gursky-Yang. When $k = n/2$, $n > 4$, some existence results were obtained by Viaclovsky and others. The cases $1 < k < n/2$ have proved much more difficult; only recently have some general results been obtained by Ge and Wang when $k = 2$ and in somewhat greater generality by Shen-Trudinger-X. Wang. All of this has provided an excellent testing ground for older techniques as well as for the development of new ones in the field of fully non-linear elliptic equations.

This now covers many (but not all) of the bases in this corner of conformal geometry. We finally come to the book under review. The author, Alice Chang, has been a key player and the leader in this recent development; the credit for uncovering these many relationships between Polyakov formulæ for determinants of conformally covariant operators and this family of new curvature quantities belongs to her and her collaborators.

As the author states in her introduction, this book should be regarded as a snapshot at a particular moment in the history of this subject. Its focus is on this circle of ideas in dimensions 2 and 4. The early chapters contain a careful account of these problems and some related ones on surfaces. In particular, she discusses some existence results for the prescribed Gauss curvature equation on surfaces and the various borderline Sobolev inequalities, including the Moser-Trudinger and Onofri inequalities, needed to carry out the rigorous proof of the existence of minimizers for the determinant functional on the 2-sphere. The original Polyakov formula is also proved. The middle set of chapters gives an overview (omitting many proofs) concerning the Paneitz operator and other conformally covariant operators and the Branson-Ørsted Polyakov formulæ. The final chapters are again quite detailed and present her proof, with Gursky and Yang, for the existence of metrics on a 4-manifold with constant σ_2. This can be recommended as a good 'case study' of an interesting fully elliptic problem.

There is a wide range of important analytic and geometric techniques used here, and though the learning curve would be steep, this book should be accessible to students after a first course in graduate PDE and some experience with Riemannian geometry (or at least its formalism), though a preliminary close reading of Lee and Parker's article on the Yamabe problem would provide context and also mitigate some of the technical demands of this book. The arguments for the two dimensional case and for σ_2 in four dimensions are explained carefully, and this provides a nice example of a beautiful interplay between geometric and analytic ideas in some accessible settings. There is no discussion of the subtleties in the prescribed Gauss curvature equation on surfaces or of the corresponding problem in higher dimensions, but I realize that the author's space and time were limited. In any case, I would certainly enthusiastically recommend this book to students or anyone wishing to learn some of these ideas.

This subject has continued to expand in the intervening years since Chang gave the Nachdiplom lectures at the ETH in Zürich, upon which these notes are based, and the understanding of some of the problems discussed here has reached a much more refined state. I must also record the very sad fact that Tom Branson, another of the principal architects of modern conformal geometry, passed away suddenly in the spring of 2006.

I conclude with a few further comments. A guiding problem in Riemannian geometry is to find on a given manifold a 'best metric', the existence of which should hopefully provide new ways to understand the topology of the manifold. Einstein metrics are the best candidates for such canonical metrics; these are characterized by the equation that their Ricci curvature is a multiple of the metric, $\mathrm{Ric}(g) = \lambda g$ for some constant λ. The direct search for Einstein metrics in higher dimensions and on general manifolds has proved far too difficult, and there are no tractable analytic approaches for proceeding (except on Kähler manifolds or otherwise in the presence of extra structure). On the other hand, as we have seen, the more restricted problem of finding good metrics within a conformal class has proved to be within the bounds of current techniques, although sometimes just barely, and these geometric problems have engendered a huge amount of progress and new analytic insight. Except in two dimensions, there is no single completely natural choice of metric in a conformal class, but instead, the solution of the Yamabe problem, and now much of this work described here, provides a collection of good conformal representatives. One of the real challenges in this subject is to discover how to use these and so repay the debt of analysis to geometry.

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