BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, April 2007, Pages 259–266
S 0273-0979(07)01157-3
Article electronically published on February 22, 2007

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
JUHA HEINONEN

MR1414889 (97j:46033) 46E35 (42B20 53C23 57N15)

Semmes, S.

Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities.

Selecta Math. (N.S.) **2** (1996), *no.* 2, 155–295.

It is a little appreciated feature of Euclidean n-space, $n \geq 2$, that every pair of points a, b in it can be joined not only by a straight line segment $[a, b]$—the shortest curve between the points—but by a large family of curves with length comparable to the distance between the points. One such family is obtained when we draw two line segments symmetrically from the points a and b to an $(n-1)$-disk of radius $|a-b|$ that meets the segment $[a, b]$ perpendicularly at its midpoint. From the existence of this curve family one deduces the important Sobolev and Poincaré inequalities that are valid in $\mathbf{R}^n$. This deduction is an abstract procedure that only uses some basic harmonic analysis, and in which the Euclidean structure no longer plays a role.

In this very interesting paper, the author addresses the issue of finding plentiful curve families joining pairs of points in a metric space. This study provides important new examples of metric spaces that support Sobolev-Poincaré inequalities similar to those in $\mathbf{R}^n$. These spaces need not be smooth in any classical sense of the word, yet for many analytic problems they are indistinguishable from Euclidean space.

It is worthwhile to describe more precisely what is meant by plentiful curve families in a metric space. Let (X, d, μ) be a (complete, separable) metric measure space; for simplicity, assume here that X has Hausdorff dimension $n > 0$ and that μ is the Hausdorff n-measure $\mathcal{H}_n$, finite on bounded sets. A family Γ of curves in X is said to be ν-controlled, where ν is a Borel measure on X, if there is a probability measure $d\gamma$ on Γ such that ν dominates the measure $A \mapsto \int_\Gamma \operatorname{length}(\gamma \cap A)\, d\gamma$, for $A \subset X$ a Borel set. Ideally, Γ should be a family of curves joining the given points a and b in X and ν should be absolutely continuous with respect to the background measure $\mu = \mathcal{H}_n$ with a nice density. If Γ is the curve family in $\mathbf{R}^n$ as described in the first paragraph, then Γ is ν-controlled with $d\nu \approx (|a-x|^{(1-n)} + |b-x|^{(1-n)})d\mathcal{H}_n$; the probability measure on Γ is an appropriately normalized angular measure. The main result of the paper gives conditions on X that guarantee the existence of a ν-controlled family Γ of curves from a point a to a point b with all curves contained in a ball of radius comparable to $d(a, b)$ and with $d\nu \approx (d(a, x)^{(1-n)} + d(b, x)^{(1-n)})d\mathcal{H}_n$.

One set of sufficient conditions given in the paper is as follows: X is an orientable topological n-manifold whose local covering and contractibility properties are uniformly controlled. I forgo the precise definitions of the last two conditions here. The first is satisfied if the Hausdorff n-measure in X is nicely distributed, say it obeys the Ahlfors regularity condition: $\mathcal{H}_n(B_R) \approx R^n$ for all balls B_R of radius $R < \operatorname{diam} X$. The second is satisfied if each ball can be contracted to a point inside a concentric ball with radius enlarged by a fixed factor. These conditions are already quite general (yet more general ones are given in the paper), leading to significant applications. However, the author is careful in pointing out that his methods are quite flexible and that there is room for weakening the assumptions while retaining the same conclusions. It would be a nice task to find such weaker assumptions in an elegant, concise formulation.

There would have been a good alternative for the (purely qualitative) assumption that X be a manifold, for this property is never really used in the paper. In all the arguments, one only needs that X has the local (co-)homological properties of an n-manifold, that is, X is "algebraically" a manifold of correct dimension (cf. Remark A.35 in the paper). This distinction is important both for conceptual reasons and for applications. Many spaces that naturally arise in mathematics, e.g. in the classical decomposition space theory or in the theory of transformation groups, cannot be expected to be manifolds although they often are so-called cohomology manifolds. These examples are relevant in this analytical context as well. For example, the author [Rev. Mat. Iberoamericana **12** (1996), no. 1, 187–275; MR1387590 (97e:57025)] showed recently that many well-known spaces of the classical decomposition space theory can be realized geometrically so as to satisfy the hypotheses required by this paper.

The curve family Γ joining the fixed points a and b is constructed from the fibers of an appropriate Lipschitz map F from the product space $X \times (0, d(a,b))$ to the n-sphere. The map $F_t = F(\cdot, t)$ should be constant outside a ball $B(a, Cd(a,b))$ for t near 0, and outside a ball $B(b, Cd(a,b))$ for t near $d(a,b)$; moreover, none of the maps F_t should be homotopic to a constant. The required Lipschitz bound is: $\operatorname{Lip} F|_{X\times(\epsilon,d(a,b)-\epsilon)} \le C\epsilon^{-1}$. The fibers of F are projected to X, and under the Lipschitz conditions, the co-area formula essentially guarantees that a desired curve family Γ can be found inside the image of the projection.

The main task of the paper is to construct the Lipschitz map described above. This is a beautiful, albeit lengthy, story, and the reader is rewarded by numerous interesting details and techniques, many important in their own right and certain to be useful in other contexts. The three assumptions on X mentioned above are used roughly speaking as follows. The controlled covering function allows one to reduce the problem to the case where X lies in some large-dimensional Euclidean space, or to assure that X is well approximated by polyhedra in the spirit of the classical dimension theory; an embedding theorem of P. Assouad [Bull. Soc. Math. France **111** (1983), no. 4, 429–448; MR0763553 (86f:54050)] is put to use here in an interesting manner. After this, the controlled contractibility is used in a familiar way together with the polyhedral structure of the surrounding or approximating rectilinear space, for maps from polyhedra into (locally) uniformly contractible spaces easily allow for quantitative extensions. The local homological conditions are used to secure a degree theory which then can be used to assure that the obtained maps are homotopically nontrivial.

Arguments of this type were used earlier by G. David and the author [in *Analysis and partial differential equations*, 101–111, Dekker, New York, 1990; MR1044784 (91c:30037)] in the special case of a quasisymmetrically deformed $\mathbf{R}^n$. In a companion paper the author [Publ. Mat. **40** (1996), no. 2, 411–430; MR1425628 (97j:46034); see the following review] discusses anew this important special case, where a more straightforward approach is possible.

The analytic inequalities that are derived from these considerations have to be reinterpreted, for there is no underlying smooth structure on X. One can define a generalized gradient of a continuous function u in a metric space to be a nonnegative Borel function g such that $|u(x) - u(y)| \leq \int_\gamma g$ for each pair of points x and y in X and for each rectifiable curve γ joining these points. Various Poincaré-type inequalities can be formulated in terms of generalized gradients, and their validity is implied by the existence of a thick family of curves between pairs of points as discussed above. The procedure is familiar from $\mathbf{R}^n$: first one averages the inequality $|u(x) - u(y)| \leq \int_\gamma g$ over all curves in the family, and then over the points x and y. (I am brushing aside here some interesting technical points related to weak- versus strong-type estimates for function/gradient pairs; cf. Appendix B in the paper under review.)

Of late, several authors have isolated the validity of a Poincaré inequality in a metric space as an important sufficient condition which leads to nontrivial analytic statements without a priori smoothness. The assumptions usually are accompanied by certain mass bounds or minor topological stipulations. For example, Poincaré inequalities without smooth structure play an important role in the theory of quasiconformal mappings, in potential theory on manifolds, and in the study of limit spaces of Riemannian manifolds with curvature bounds. The paper by Semmes is necessary reading for anyone interested in this type of geometric analysis. The reader should not fear the daunting length of the paper, much of which is caused by extremely careful exposition.

(From MathSciNet, November 2006)

Juha Heinonen

MR1699320 (2000d:53065) 53C23 (53-02)

Gromov, Misha

Metric structures for Riemannian and non-Riemannian spaces. (English summary)

Based on the 1981 French original [MR0682063 (85e:53051)].
With appendices by M. Katz, P. Pansu and S. Semmes.
Translated from the French by Sean Michael Bates.
Progress in Mathematics, 152.
Birkhäuser Boston, Inc., *Boston, MA*, 1999. *xx*+585 *pp*. $89.95.
ISBN 0-8176-3898-9

The first edition of this book, published in French [*Structures métriques pour les variétés riemanniennes*, Edited by J. Lafontaine and P. Pansu, CEDIC, Paris, 1981; MR0682063 (85e:53051)], is considered one of the most influential books in geometry in the last twenty years. Since then the boundary of the field has dramatically exploded. Reflecting this growth, the new English edition has almost quadrupled in size.

Among the most substantial additions, each taking over a hundred pages, there is a chapter on convergence of metric spaces with measures, and an appendix on analysis on metric spaces written by Semmes. In addition, numerous remarks, examples, proofs, and open problems are inserted throughout the book. The original text is mostly preserved with new items conveniently indicated by a subscript +. We now describe the contents in more detail.

The first chapter contains general results on path metric spaces with numerous examples that often go beyond the Riemannian category. Chapter 2 discusses invariants of maps, such as degree and dilation. Chapter 3 is devoted to convergence of metric spaces, starting from Lipschitz and Hausdorff convergence and their generalizations, such as ultraproducts and convergence with control.

The next chapter, labeled 3.5, is entirely new. It is devoted to convergence of metric spaces with measures and the Lévy-Milman concentration phenomenon. Chapter 4 deals with norms on homology and systoles and contains two new sections on unstable systolic inequalities, filling, and systoles of universal spaces. This theme is continued in the appendix on systolically free manifolds written by Katz.

Chapter 5 discusses manifolds with lower Ricci curvature bound. This area has seen a remarkable growth in the last few years. Accordingly, five new sections are added dealing with, among other things, packing inequalities, simplicial norms, L^2-Betti numbers, and entropy. Chapter 6 is about isoperimetric inequalities with a new section on Varopoulos' isoperimetric inequality. Chapter 7 discusses Morse theory, minimal models, and bounds on dilation.

Chapter 8 contains a discussion of pinching and collapse. This chapter has been rewritten entirely even though some of the material (e.g. the Cheeger-Gromov compactness theorem) is borrowed from the French original. References are given to more recent work on collapse under lower curvature bounds.

There are four appendices: on "quasiconvex" domains written by Pansu, on systolic freedom by Katz, and on Lévy's isoperimetric inequality by Gromov. Finally, an appendix written by Semmes is devoted to analysis on metric spaces with doubling measures (such as Heisenberg group or Sierpiński gasket). In Gromov's words: "here, the reader can painlessly learn several key ideas of real analysis made accessible to us geometers by the masterful exposition of Stephen Semmes, who has adapted his material to our non-analytic minds".

This book is certain to be a source of inspiration for many researchers as well as the required reading for students entering the subject.

(From MathSciNet, November 2006)

Igor Belegradek

MR1708448 (2000g:53043) 53C23 (49J52)

Cheeger, J.

Differentiability of Lipschitz functions on metric measure spaces.

Geom. Funct. Anal. **9** (1999), *no.* 3, 428–517.

This paper extends fundamental notions from the calculus of first derivatives to very general classes of metric measure spaces. A notable example is a generalization of Rademacher's classical theorem on almost everywhere differentiability of Lipschitz functions. This work has already had a major impact on the subject, and these results, as well as the arguments developed to prove them, will surely have significant applications.

In recent years, there has been a great deal of progress on "low regularity" ideas in various contexts (many people have made contributions [cf., e.g., R. R. Coifman and G. Weiss, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645; MR0447954 (56 #6264); G. David and S. W. Semmes, *Analysis of and on uniformly rectifiable sets*, Amer. Math. Soc., Providence, RI, 1993; MR1251061 (94i:28003); S. W.

Semmes, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, 875–885, Birkhäuser, Basel, 1995; MR1403987 (97j:28010); J. Heinonen and P. Koskela, Acta Math. **181** (1998), no. 1, 1–61; MR1654771 (99j:30025); and references therein]). These spaces also arise naturally in classical geometric contexts; for instance, as the (measured) Gromov-Hausdorff limits studied by K. Fukaya in [Invent. Math. **87** (1987), no. 3, 517–547; MR0874035 (88d:58125)] and Cheeger and T. H. Colding in [C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 3, 353–357; MR1320384 (96f:53047)] and the papers announced therein.

A function f on a metric space Z is Lipschitz (with norm C) if $|f(z_1) - f(z_2)| \leq C\,\mathrm{dist}_Z(z_1, z_2)$ for all $z_1, z_2 \in Z$. Specifying a measure μ on Z, we can define the L^p spaces. The natural questions are then: What are tangent spaces to $z \in Z$? Derivatives? Sobolev spaces $H^{1,p}$?

After a preliminary definition, we will survey some of the results of the paper. Heinonen and Koskela say that g is an upper gradient for f if, for all $z_1, z_2 \in Z$ and rectifiable curves $c\colon [0,l] \to Z$ (parametrized by arclength) from z_1 to z_2,

$$|f(z_1) - f(z_2)| \leq \int_0^l g(c(s))\,ds.$$

Clearly, the Lipschitz norm is an upper gradient. If $f \in L^p$ has a (generalized) upper gradient $g \in L^p$, then $f \in H^{1,p}$. Section 2 then develops the basic theory; e.g., §2.10 shows the existence of a minimal generalized upper gradient g_f. $f \in H^{1,p}$ is then p-harmonic if it minimizes the L^p norm for its g_f; it is asymptotically p-harmonic at z if this is true on average over balls of radius $r \to 0$ about z. In §3.5, f is defined to be asymptotically generalized linear (or "agl") with respect to g_f at $z \in Z$ if f is asymptotically p-harmonic at z and z is a Lebesgue point of g_f^p. Since linear functions on $\mathbf{R}^n$ are p-harmonic (any p), it follows easily that $f \in C^1(\mathbf{R}^n)$ is agl everywhere. Theorem 3.7 then states: If Z, μ satisfies the Vitali covering theorem and f is Lipschitz, then f is agl μ-a.e. This generalizes half of Rademacher's theorem; it remains to get some uniqueness of the agl structure on different scales (this is done in later sections).

Many of the results from Section 4 on assume that μ is doubling (which implies the Vitali covering theorem) and that Z, μ has certain Poincaré inequalities. For simplicity, we assume this for the remainder of this review; unfortunately, this means that some things will be oversimplified below.

In Section 4, the author obtains a uniqueness result for his generalized Rademacher theorems (4.38). This has strong consequences, including existence and uniqueness of strong derivatives (4.47), reflexivity of the $H^{1,p}$ spaces (4.48), and the construction of a finite-dimensional generalized cotangent bundle for Z; this basic structure plays a key role in Sections 5 and 6. Those sections characterize the minimal upper gradient g_f (in terms of a local Lipschitz constant; §6.1). In Section 7, the author studies p-harmonic functions, proving a maximum principle and solving the Dirichlet problem in this general setting.

Fix $1 < p < \infty$ and Z, μ (with the doubling and Poincaré). A Lipschitz l is generalized linear ("gl") if (1) either $l \equiv 0$ or its range is $(-\infty, \infty)$, (2) l is p-harmonic, and (3) g_l is constant. The author obtains a representation theorem for these and a generalized splitting theorem (8.5 and 8.11) [cf. J. Cheeger and D. Gromoll, J. Differential Geometry **6** (1971/72), 119–128; MR0303460 (46 #2597)]. If $Z = \mathbf{R}^n$ and $\mu = \mathcal{H}^n$, then l is linear by 8.11. Section 9 shows that the doubling and Poincaré persist under suitable limits [cf. K. Fukaya, op. cit.; MR0874035

(88d:58125); J. Cheeger and T. H. Colding, op. cit.; MR1320384 (96f:53047)]. Section 10 strengthens the uniqueness statement and completes the proof of the generalized Rademacher theorem (in 10.2); this uses much of the theory developed in earlier sections. Section 11 focuses on small-scale and infinitesimal regularity of Z, μ and, in particular, obtains almost everywhere "regularity" for *some* tangent cones (11.7). Section 12 investigates the structure of the generalized cotangent bundle of Z, μ (and the natural norm on it), and Section 13 continues this, bringing in Hausdorff measures (one consequence relates the tangent cones to the tangent space, i.e., the dual space of the cotangent bundle; see 13.4).

Section 14 asks when Z, μ admits a bi-Lipschitz imbedding into $\mathbf{R}^N$, and the main result (14.3) gives a necessary condition (roughly, almost everywhere, tangent cones must be bi-Lipschitz to a Euclidean space of dimension determined by 4.38). This implies that Carnot-Carathéodory spaces, Laakso spaces, and boundaries of two-dimensional hyperbolic buildings all do not admit bi-Lipschitz imbeddings into $\mathbf{R}^N$. The first two of these were previously known, the first due independently to Assouad and Semmes (relying on P. Pansu's differentiability theorem), the second due to T. Laakso. It is clear that this approach will yield more answers.

Z, μ satisfies an (ϵ, δ)-inequality (15.4) if there exists $C = C_{\epsilon,\delta}$ so that given $z_1, z_2 \in Z$ there exist z_1', z_2' with $|z_i - z_i'| \leq \delta\, |z_1 - z_2|$ and

$$\inf_c \int_c g(c(s))\, ds \leq C\, |z_1 - z_2| \int_{B_{(1+2\delta)(1+\epsilon)|z_1-z_2|}(z_1)} g\, d\mu,$$

where c is a curve from z_1 to z_2 whose length is at most $(1+\epsilon)\, |z_1 - z_2|$. Section 15 shows that a "thickly minimally connected" length space has such inequalities. This very general class of spaces covers all the earlier examples. These results are exploited in Section 16 to obtain some quantitative analogs of the earlier results on a small (but definite) scale. These quantitative results have significant applications in the context of [J. Cheeger and T. H. Colding, "On the structure of spaces with Ricci curvature bounded below. III", Preprint; per bibl.; J. Cheeger, T. H. Colding and W. P. Minicozzi, II, Geom. Funct. Anal. **5** (1995), no. 6, 948–954; MR1361516 (96j:53038)]. One consequence (of 16.32) is then that on $\mathbf{R}^n$ agl functions converge to linear functions as $r \to 0$.

(From MathSciNet, November 2006)

William P. Minicozzi, II

MR1930885 (2004k:53057) 53C24 (30C65 51F99 57M99)

Bonk, Mario; Kleiner, Bruce

Quasisymmetric parametrizations of two-dimensional metric spheres.

Invent. Math. **150** (2002), *no.* 1, 127–183.

This long and thoughtfully-written paper considers a "uniformization problem" in the context of two-dimensional compact metric spaces. The classical uniformization theorem implies (among other things) that a Riemann surface homeomorphic to the 2-sphere $\mathbb{S}^2$ is conformally equivalent to it. Since conformality cannot be an issue in general metric spaces, the authors transfer the problem to a situation to which the machinery of geometric metric space analysis (standard reference: [J. Heinonen, *Lectures on analysis on metric spaces*, Springer, New York, 2001; MR1800917 (2002c:30028)]) may be applied in a very natural way. In particular, a reader will obtain as dividend an insight into the impact and methodology of this type of analysis. At the end of the introduction, the authors present a strategy for the reader less familiar with these techniques to see the main directions, in particular the intricate arguments using the modulus of curve families; the modulus is the key analytic weapon used here.

The object of study is a metric space (Z, d) which is assumed to be homeomorphic to $\mathbb{S}^2$. The problem is to determine necessary and sufficient (intrinsic) conditions on (Z, d) so that there will exist a quasisymmetric homeomorphism $f\colon Z \to \mathbb{S}^2$; recall that a homeomorphism $f\colon X \to Y$ between metric spaces X and Y is quasisymmetric (qs) if there is an orientation-preserving homeomorphism η of $(0, \infty)$ such that

$$\frac{d_Y(f(x_1), f(x_2))}{d_Y(f(x_1), f(x_3))} \leq \eta\left(\frac{d_X(x_1, x_2)}{d_X(x_1, x_3)}\right)$$

for every triple $\{x_i\} \in X$.

The statement of the precise equivalent conditions is deferred to §11, since these depend on technical notions introduced in the proof, and seem unwieldy to verify in practice. However, the authors present simpler (but nearly equivalent) versions which give sufficient conditions for Z to be qs equivalent to $\mathbb{S}^2$.

If (Z, d) is qs equivalent to $\mathbb{S}^2$, then it is immediate that Z itself must be linearly locally connected (llc) and a metric doubling space; we call these the elementary properties. However, these alone cannot be sufficient; for example one could impose a metric d_l on $\mathbb{S}^2$ where $d_l = \sigma^l$, with σ spherical distance and $l \neq 1$. The most appealing answer, presented as Theorem 1.1, is that Z must in addition be Ahlfors Q-regular with $Q = 2$, so that the Hausdorff 2-measure of each $B(x, r)$ is comparable to r^2. This also provides an affirmative answer to a question from [J. Heinonen and S. Semmes, Conform. Geom. Dyn. **1** (1997), 1–12 (electronic); MR1452413 (99h:28012)].

The proof that, for example, Ahlfors 2-regularity, together with the elementary conditions, is sufficient, requires construction of the desired qs map f. The starting point is to mine elementary consequences of Z fulfilling these elementary properties. In particular, for each $r > 0$ the fact that Z is llc and doubling allows the authors to embed a graph G into Z so that G shares properties analogous to those of the standard tilings of the plane and the sphere into cells of sidelength comparable to r. The 1-skeleton of G may be used to provide a triangulation of Z, and then the Andreev-Koebe-Thurston theorem automatically transfers this triangulation to one on $\mathbb{S}^2$, with an associated circle packing. Thus the first concrete step toward obtaining f is the map which sends vertices of G to the centers of the circle packing on $\mathbb{S}^2$. The authors introduce finer "graph approximations" G to Z. In order to obtain mappings which converge to the desired qs map f, the authors move back and forth from G to Z, transferring these maps defined on the graph approximations to maps from Z to $\mathbb{S}^2$. To show that the process converges to a qs map as $r \to 0$, the modulus must be exploited. For Z, the modulus is always available since Z is Q-Loewner (this works for any $Q \geq 1$), but for G the modulus is introduced and developed in §7. The 2-regularity of Z is needed in §9 to show that the approximating maps $Z \to \mathbb{S}^2$ satisfy uniform modulus estimates. While the details are intricate, they are presented well.

These results are decidedly two-dimensional; even in the introduction the authors discuss counterexamples for $n > 2$, and note that qs, rather than bi-Lipschitz, seems to be the appropriate category of maps to consider (an open question about bi-Lipschitz maps is posed at the end of the paper, which is connected to the "Jacobian problem" for maps on $\mathbb{S}^2$).

The orientation of this paper is similar to that of several others, in particular [J. W. Cannon, Acta Math. **173** (1994), no. 2, 155–234; MR1301392 (95k:30046)]. Cannon's focus is on a topological space Z (rather than a metric space), but his reliance on a combinatorial version of the modulus is considered by the authors to be "similar in spirit" to the analysis used here. Some questions concerning the boundary $\partial_\infty G$ of Gromov hyperbolic groups G are proposed, in particular whether $\partial_\infty G$ can be taken qs equivalent to spaces with simple analytic properties, for example to spaces for which Poincaré inequalities hold. The authors also note that Theorem 1.1 may be rephrased to provide a condition for a complete 2-regular metric space to be qs equivalent to $\mathbb{R}^2$. The paper concludes with an example of a metric space of Hausdorff dimension greater than two which is qs-equivalent to $\mathbb{S}^2$.

(From MathSciNet, November 2006)

D. Drasin