BOOK REVIEWS

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Orthogonal polynomials on the unit circle, Parts 1 and 2, by Barry Simon, Amer. Math. Soc., Providence, RI, 2005, xxvi+466 pp., US$89.00, ISBN 978-0-8218-3446-6 (Part 1); xxii+578 pp., US$99.00, ISBN 978-0-8218-3675-0 (Part 2); US$149.00, ISBN 978-0-8218-3757-3 (set)

"...and there was (spectral) light..."[1]

NOTE. If the reader is interested in reading a comprehensive, magnificently written, and up-to-date history of orthogonal polynomials, I recommend turning to L. Golinskii and V. Totik [8] and Totik [30]. On the other hand, if the reader wants to see a fairly detailed but not overly technical review of Barry Simon's book, then I suggest Simon's article [24] despite the obvious conflict of interest. End of story, or at least it seems that way. However, my editor insisted that the story must go on, which explains why I agreed to write this review, which, when compared to [8], [30], and [24], is doomed to fail.

If you are still with me, then let's get on with the definitions. Given a non-negative Borel measure μ on the unit circle $\mathbb{T}$ with infinite support, orthogonal polynomials (**OPs**) on the unit circle (**OPUC**) are polynomials $(\varphi_n(\mu))_{n=0}^{\infty}$ of precise degree n that are orthonormal with respect to the inner product

$$(f, g) = \frac{1}{2\pi}\int_{\mathbb{T}} f\overline{g}d\mu\,.$$

In addition, Φ_n denotes the monic version of φ_n. On the other hand, OPs on the real line (**OPRL**) are analogous thingies where the measure lives on $\mathbb{R}$.[2]

OPs, both on $\mathbb{R}$ and $\mathbb{T}$, satisfy some type of a recurrence relation which, one way or another, leads either to a continued fraction or a (multiplication) operator. Hence, the name of the game is simple. Given either some measure or OPs or a recurrence relation or moments or a continued fraction or a (multiplication) operator or a related item such as a Stieltjes transform or a Jacobi (banded, Hessenberg) matrix or a Carathéodory function or a Schur function or any combination of the above, one needs to find out as much as possible about the rest of the gang.

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[1]Alphonse P. Magnus à la Alexander Pope; see http://en.wikipedia.org/wiki/Alexander_Pope.

[2]OPs, OPUC, OPRL, and such are used both as singular and plural, depending on the context.

Here are four elegant and striking results obtained in the past 30 some years,[3] which can be understood and even perhaps appreciated by anyone who read the second paragraph of this review. Namely, if (i) $\mu' > 0$ a.e., then (ii) $(\Phi_n(\mu,0))_{n=0}^{\infty}$ converges to 0, but (iii) there are both pure point-mass and singularly continuous measures μ for which (ii) still holds even though (i) and (ii) are almost equivalent conditions in the sense that (iv)

$$\mu' > 0 \quad \text{a.e.} \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \sup_{\ell\geq 1} \int_0^{2\pi} \left| \frac{|\varphi_n(\mu,e^{it})|^2}{|\varphi_{n+\ell}(\mu,e^{it})|^2} - 1 \right| dt = 0$$

and

$$\lim_{n\to\infty} \Phi_n(\mu,0) = 0 \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \inf_{\ell\geq 1} \int_0^{2\pi} \left| \frac{|\varphi_n(\mu,e^{it})|^2}{|\varphi_{n+\ell}(\mu,e^{it})|^2} - 1 \right| dt = 0\,.$$

Now let's proceed with the actual review of Barry Simon's writings [21, 22, 23]. According to tradition, BAMS book reviews serve multiple purposes.

> First, they are supposed to give an overview of the subject area covered by the book under consideration.
> Second, they are supposed to discuss the actual contents of the book.
> Third, they are supposed to expose the strengths and weaknesses of the book in a constructive, preferably amicable but critical manner.
> Fourth, they are supposed to demonstrate the extraordinary wittiness and infinite intelligence of the review–writer himself.

We can easily dispose of the fourth item on the list above since it is well known that the writer of this particular review is neither witty nor intelligent, so that it would be a waste of time to try to convince the reader to the contrary.

So what the heck are OPs? It is an area of real and complex analysis, I mean harmonic analysis, I mean algebra, I mean combinatorics, I mean differential equations, I mean special functions, I mean representation theory, I mean numerical analysis, I mean mathematical physics, I mean partition theory, I mean coding theory, I mean probability theory, I mean number theory, I mean quantum groups, I mean applied mathematics,... **STOP**, this is becoming too confusing, so let's start all over again.

Barry Simon is mostly (but not exclusively) interested in OPs as a subject belonging to some version of general analysis as opposed to some of the other subject areas listed above.

OPs is a mathematical subject whose history could be best demonstrated by the Chebyshev polynomial of nth degree associated with the interval $[-1,1]$, that is, by $\cos nt$ where $x = \cos t$ for $-1 \leq x \leq 1$. In other words, it has had its own share of ups and downs. As of 2006, I would say that n is in the neighborhood of 9. The ups are characterized by introducing new techniques, fresh points of views, and by finding new applications, whereas the downs designate periods of stagnation when the old ideas are exhausted and the new ones have not yet been injected.[4]

Here is a very personal, very one-sided, and very arguable history of OPs.

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[3] I picked these results for illustration since (i) their proofs are genuinely sophisticated, and (ii) they involve almost two dozen or so OPs people, a majority of whom will not get mad at me for not mentioning their names.

[4] Clearly, n is even $\Longleftrightarrow$ G.d exists.

> brute force $\Longrightarrow$ special functions $\Longrightarrow$ real analysis $\Longrightarrow$ complex analysis $\Longrightarrow$ continued fractions $\Longrightarrow$ linear algebra $\Longrightarrow$ harmonic analysis $\Longrightarrow$ operator theory $\Longrightarrow$ scattering theory $\Longrightarrow$ difference equations $\Longrightarrow$ potential theory $\Longrightarrow$ matrix theory $\Longrightarrow$ Lax–Levermore theory $\Longrightarrow$ Riemann–Hilbert methods $\Longrightarrow$ spectral analysis

Of course, there is a huge overlap, mixing, and multiplicity.

As far as I am concerned, the theory of OPs started in 1814,[5] when Johann Carl Friedrich Gauss published [5], where he introduced a special case of what ended up being called Gauss–Jacobi quadrature.[6] Gauss himself used continued fractions and didn't relate his quadrature formula to OPs. The latter connection was found by Carl Gustav Jacob Jacobi [10], and, therefore, most experts would probably start the history of OPs with Jacobi.[7] Still the alea had been iacta in 1814.[8] Since then, there have been long periods when interest in OPs was either practically non-existent or, to the contrary, it attracted some of the best mathematical minds.

In a sense it can be compared to the history of Fourier series. For instance, anyone who ever studied pointwise or uniform convergence of trigonometric Fourier series remembers that guys such as Riemann, Lebesgue, Cantor, Dini, and Lipschitz spent a considerable amount of effort on proving various convergence criteria that drew a bored "so what" for a long time in the 20th century until Lennart Carleson woke up the analysts in 1965, and then convergence again became a hot topic until all figured out that they can't find a simple proof of Carleson's theorem.[9]

The same can be said of OPs. I am not a mathematics historian, but let me try to inject a few names here in some kind of chronological order:

L. Lagrange (1736–1813), P. S. Laplace (1749–1827), A. M. Legendre (1752–1833), C. F. Gauss (1777–1855), C. G. J. Jacobi (1804–1851), P. L. Chebyshev (1821–1894),[10] C. Hermite (1822–1901), E. Laguerre (1834–1886), T. Stieltjes (1856–1894), A. A. Markov (1856–1922), H. Poincaré (1854–1912), F. Hausdorff (1868–1942), L. Fejér (1880–1959), S. Bernstein (1880–1968), M. Riesz (1886–1969), V. I. Smirnov (1887–1974), G. Pólya (1887–1985), H. L. Hamburger (1889–1956),[11] J. L. Walsh (1895–1973), G. Szegő(1895–1985), Ya. L. Geronimus (1898–1984),[12] A. Zygmund (1900–1992), N. I. Akhiezer (1901–1980), A. N. Kolmogorov (1903–1987), M. Stone (1903–1989), M. G. Krein (1907–1989), P. Turán (1910–1976), P. Erdős (1913–1996), and G. Freud (1922–1979).[13]

[5]So OPs start when Napoleon ends. However, there is a much more straightforward and obvious connection between Napoleon and OPUC. Namely, Napoleon $\Rightarrow$ *War & Peace* $\Rightarrow$ Tolstoy $\Rightarrow$ Tolstoy's year of birth $\Rightarrow e \stackrel{\text{def}}{=} 2.718281828\ldots \Rightarrow$ Euler $\Rightarrow$ [22, Theorem 11.9.1, p. 790].

[6]See, e.g., `http://mathworld.wolfram.com/GaussianQuadrature.html`.

[7]Of course, googling "legendre polynomial 1782" shows that the task of finding the first OPs guy is not exactly trivial.

[8]See, e.g., `http://www.infoplease.com/askeds/6-6-01askeds.html`.

[9]For a comprehensive account, see, e.g., [11].

[10]Let me plug here the not-so-widely-read but superbly entertaining *The Thread: A Mathematical Yarn* by P. J. Davis, Birkhäuser, 1983.

[11]Not in [14]; use "hans ludwig hamburger" to google him, only 14 hits in May 2006; for a photo see [9, p. 24].

[12]Not in [14]; for details see [7, 19].

[13]Not in [14]; he was born on 01/04/1922 in Budapest, Hungary, as a Jew, and he died on 09/27/1979 in Columbus, Ohio, USA, as a Catholic. For a photo and details see [16] or google "geza freud".

It is fair to say that if, in some cases indirectly, OPs were good enough for these gentlemen,[14] then they should be good enough for the rest of us.

OPUC are a small part of a general theory which probably have more order and more intrinsic beauty in them than the rest of the theory combined, mostly due to the wonderful properties of the unit disk and the unit circle that are well known to all who have ever studied complex analysis.

In particular, the simple property that conjugation and taking reciprocals on the unit circle are the same operations leads to some amazing discoveries, first observed probably by Peter Lax's uncle,[15] who introduced OPUC in [27, p. 153] and started their intensive study in [27, pp. 235 and 277], that various seemingly different extremal problems are, de facto, the same studied from different points of view.

Example. Given $0 < p < \infty$ and a non-negative Borel measure μ on the unit circle $\mathbb{T}$, the least L^p_μ "norm" of nth degree monic polynomials is the same as the least L^p_μ "norm" of nth degree polynomials that take the value 1 at the origin.

Denoting the above least "norm" by $E_n(p)$, clearly, $(E_n(p))_{n=0}^\infty$ is a decreasing non-negative sequence and, hence, has a limit, say $E(p)$. Using Jensen's formula or Jensen's inequality, one can easily obtain a lower bound for $E(p)$ in terms of the geometric average of the absolutely continuous component of the measure μ. It was Szegő who proved that the obvious lower bound is, in fact, the limit.[16]

For the sake of fairness, let's start the countdown of the history of the theory OPUC from 1912 or 1913, when, according to George (then still György) Pólya, he told Szegő that he found the limit of the nth root of certain $n \times n$ Toeplitz determinants, say D_n, except that he didn't prove it. Szegő not only proved it but he improved it by showing that, in fact, the limit of D_n/D_{n-1} exists (which is essentially the same as the limit of $E_n(2)$), and, thereby, Szegő became the founding father of OPUC.

Pólya wrote in [27, p. 11] the following:

> Our cooperation started from a conjecture which I found. It was about a determinant considered by Toeplitz and others, formed with the Fourier coefficients of a function $f(x)$. I had no proof, but I published the conjecture[17] and the young Szegő found the proof and published it in the Mathematische Annalen [27, p. 53]. This was his first published paper.

For the sake of *unfairness*, let me point out that Szegő gives credit to M. Fekete[18] for studying the limit of D_n/D_{n-1}.[19] As we all know, Fekete eventually became the transfinite diameter guy,[20] so even though he does not get credit for OPUC,

[14] Only departed people are listed, so, e.g., no Gel'fand, no Sarnak, no Askey, no Totik, no Khrushchev, and definitely no Simon.

[15] AKA Gábor Szegő.

[16] See, e.g., U. Grenander and Szegő's book [22, item 479, Section 3.1, and the notes on p. 231] for this approach. Originally, Szegő considered weight functions only. Extensions to general measures came later. Blame the Soviets.

[17] See the third footnote in [27, p. 55] and the footnote on p. 81.

[18] Not in [14]; for photos see [20, pp. 115, 133, and 134].

[19] See the top of [27, p. 56] and the second footnote on p. 82.

[20] See `http://eom.springer.de/T/t093670.htm`, in particular, [22, item 346] and [3].

many years later, according to Hegelian rules of dialectic,[21] the two concepts finally did rendezvous.

As far as I know, Szegő's numerous papers initially didn't draw much attention, not even among his friends and colleagues. The OPRL people didn't seem to care about OPUC, and those few who studied OPUC were themselves working in a vacuum.

For instance, up until Barry Simon's relentless search for the truth, no one seemed to have cared for Samuel Verblunsky,[22] who all by himself discovered numerous extraordinarily important facts about OPUC; see [22, items 1066–1070].[23]

Another example is J. A. Shohat,[24] whose work and even book titled *Théorie Générale des Polinomes Orthogonaux de Tchebichef* [22, item 957] were more or less ignored.

Yet another example is the case of the *Levinson Algorithm*. According to the story I heard, Norman Levinson[25] was unaware of Szegő's work when he introduced it in 1947; see [22, item 698].[26]

The only exception I can think of is a very distinguished group of Soviet mathematicians such as N. I. Akhiezer, S. N. Bernstein, Ya. L. Geronimus, A. N. Kolmogorov, M. G. Krein, and V. I. Smirnov, who not only recognized the significance of Szegő's work but also extended it in various directions.[27]

Even Barry Simon seems to agree with this gloomy assessment; see, e.g., [21, p. 9], where Barry writes the following:

> While OPRL has many fathers, Szegő dominated the early work on OPUC. Indeed, except for the work of Akhiezer–Krein on the trigonometric problem and of Verblunsky,. . . , Szegő alone studied these problems during their first twenty years until the extensive work of Geronimus and his school starting in the 1940's.

Szegő, on the other hand, kept himself well informed of the work done by both the Soviet people working in OPs and of the isolated instances of contributions by others such as S. Verblunsky and F. Pollaczek.[28] It is an ironic act of fate that even one of Szegő's students, Albert Boris J. Novikoff, was of Russian background. Although Novikoff never published anything on OPs, his doctoral dissertation [18] on Pollaczek polynomials played a constructive role in the development of OPUC in the last fifth of the 20th century.

After the publication of Szegő's immensely popular book on OPs [26], which, as of 2006, had four editions and eleven printings,[29] and his joint book with U. Grenander [22, item 479], the situation started to change somewhat in the 1960's, especially

[21]See, e.g., `http://en.wikipedia.org/wiki/Dialectic.`

[22]He was neither Polish nor Russian, but English, a student of J. E. Littlewood; see `http://genealogy.math.ndsu.nodak.edu/html/id.phtml?id=18561.`

[23]Some Russians such as Ya. L. Geronimus and his students, including my good friend Lenya Golinskii's dad, Boris L. Golinskii, knew of Verblunsky's work. While I was writing this review, Lenya's dad died on 5/12/2006 in Kharkov, Ukraine.

[24]AKA Jacques Chokhate, not French but a Russian/American Jew.

[25]See [14, Mathematicians/Levinson.html].

[26]MR review by J. L. Doob; see MR0019257 on MathSciNet.

[27]See [12] for an entertaining and scholarly account of Soviet mathematical life during that era.

[28]See MR1672417 and MR0633245 on MathSciNet for a brief and sad biography. As of May of 2006, his first wife's second husband got over 35K hits on google.

[29][26] was published in the same AMS Colloquium Series as Simon's book.

in the Soviet Union. Even then, important papers on Toeplitz matrices, such as the ones by H. Widom [22, item 1094] in 1965, and on OPUC, such as the ones by G. Baxter [22, item 92] in 1961 and by P. Delsarte, Y. Genin, and Y. Kamp's [22, item 260] in 1978 were largely ignored, at least within the OPs community.

OPUC started to become both respected and popular in the mid-1970's, when two events took place independently of each other.

First, K. M. Case wrote a number of articles in *J. Math. Phys.* in 1974 which (i) probably no person interested in OPs read at the time except possibly his own students, (ii) had numerous problems in terms of accuracy of statements and correctness of proofs, and (iii) culminated in the survey paper [2].

Although Case's approach to OPs via scattering theory was not new, nor did he claim credit for it, the primary advantage of it was that [2] was published in a conference proceedings edited by a relentless promoter of the area, Dick Askey, so that Case's paper ended up being read by a few people who were ready to start to look at OPs in general, and at OPUC in particular, from a fresh point of view. The basic idea is to study OPs using the recurrence formulas and difference equations satisfied by them. I will not try to guess where these ideas are coming from, but it can't hurt to mention the names of I. M. Gel'fand and B. M. Levitan; see, e.g., [6] and [22, item 387], which were reviewed in MR by none other than Norman Levinson.[30]

A typical result by Case would claim that if the recurrence coefficients converge fast enough, then the spectral measure associated with the corresponding OPs is absolutely continuous. In other words, the properties of measure are obtained from the OPs themselves. In a sense, this is not far from H. Weyl's theorem on compact perturbations of self-adjoint operators.

Second, E. A. Rakhmanov [22, item 885] came up with, what else, *Rakhmanov's Theorem*. Rakhmanov's theorem, 60 years after Szegő's initial results, gave the first novel theorem regarding OPUC. Namely, Szegő proved that the sequence of monic OPUC evaluated at the origin is in ℓ^2 under certain conditions, whereas Rakhmanov proved that it converges to zero under certain weaker conditions. Szegő's condition was what is nowadays called *Szegő's condition*, which requires that the orthogonality measure μ satisfy $\log \mu' \in L^1$, and Rakhmanov needed the *Erdős condition* that $\mu' > 0$ almost everywhere.[31]

Initially, Rakhmanov's theorem was meant to be a tool for studying, together with his advisor A. A. Gonchar, some convergence properties of Padé and rational approximations. In fact, it was A. A. Gonchar (see [22, item 469, formula (11) and the paragraph following it]) who suggested that the full power of Szegő's theorem was not needed and it could be replaced by a condition that essentially stated that the Jacobi matrix associated with the OPRL be a compact perturbation of a constant Jacobi matrix.

As it turned out later, Rakhmanov's original proof was incomplete because it used a formula of Ya. L. Geronimus which contained a typo.[32]

Amazingly, neither Case nor Rakhmanov realized or anticipated the eventual impact of their papers.

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[30]See MR0043315 and MR0045281 on MathSciNet.

[31]The first non-OPs and non-approximator person guessing correctly why this condition carries Erdős' name will be awarded US$10.00.

[32]See Máté–N. [22, item 754] for the entire story, and Rakhmanov [22, item 889] and Máté–N.–Totik [22, item 757] for the first correct proofs.

The next 30 or so years led to solving numerous conjectures, creating new theories, introducing new (and old) techniques, finding new applications, and, most importantly, finding new connections between OPs and other parts of mathematics. Instead of giving a blow-by-blow account of what happened, I will describe some of the developments, biased by my own interest, which could be appreciated by the average reader of this review.

In Szegő's theory, a fundamental role is played by the *Szegő function* $D(w)$ defined by

$$D(w,z) \stackrel{\text{def}}{=} \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(w(\theta))d\theta\right), \qquad |z|<1\,.$$

Many of the results assume that $0 < D(\mu',0) < \infty$, and then they conclude that some OPUC-quantity associated with μ converges to an expression involving $D(\mu')$. The multiplicative nature of the D-function brings up the question to what extent one can relax the condition that $\log\mu' \in L^1$. This gave rise to what is usually called *comparative theory* of OPs mostly developed in the 1980's. A typical result affirms that if one measure is absolutely continuous with respect to another one, say, $d\mu_2 = g d\mu_1$, then the corresponding OPUC or OPRL behave similarly as long as $0 < D(g,0) < \infty$.

Example. Let $\mu_1' > 0$ almost everywhere, let g be a positive continuous function, and let $d\mu_2 = g d\mu_1$. Then

$$\lim_{n\to\infty}\frac{\varphi_n(\mu_1,z)}{\varphi_n(\mu_2,z)} = \overline{D}(g,1/z)\,, \qquad |z|>1\,;$$

see [22, Section 9.4].

Another example is the original proof of the Freud conjectures for weights $w(x) \stackrel{\text{def}}{=} exp(-|x|^\alpha)$ on $\mathbb{R}$, where an essential role was played by the relationship between the recurrence coefficients for OPRL and their analogues for OPUC. In short, the Freud conjectures were about the asymptotic behavior of the sequence $(E_n(w)/E_{n+1}(w))_{n=0}^\infty$, where $E_n(w)$ denotes the rate of best L_w^2 approximation of x^n by lower degree polynomials. The concentrated teamwork towards the solution of the Freud conjectures and the subsequent wide-ranging simplifications and generalizations were some of the primary forces behind the revitalization of OPRL starting with the early 1970's when my advisor, Géza Freud, was still alive and provided a Simon–like burst of energy which attracted many of us to the subject. A good starting point is either to google "freud conjectures orthogonal" or to take a look at [15], [28], and [22, item 809].

Another example involves Wall's continued fractions and Schur's algorithm, which have thoroughly been studied by S. Khrushchev, who entered the OPUC scene in the late 1990's and immediately had an impact on OPUC which is comparable or, if I am allowed to say it, more profound than that of all of us who worked in the subject prior to him with the exception of perhaps V. Totik.

Let $\mathbb{D}$ denote the unit disc, let m be the normalized Lebesgue measure on $\mathbb{T}$, and let $\mathcal{B}$ be the unit ball in the Hardy algebra $H^\infty(\mathbb{D})$. Let f be a function in $\mathcal{B}$ with Schur parameters $(a_n)_{n=0}^\infty$, and let $(A_n/B_n)_{n=0}^\infty$ be the sequence of the even convergents to the Wall continued fraction for f, that is,

$$f = a_0 + \frac{(1-|a_0|^2)z}{\overline{a}_0 z}{\atop +}\frac{1}{a_1}{\atop +}\frac{(1-|a_1|^2)z}{\overline{a}_1 z}{\atop +\cdots+}\frac{1}{a_n}{\atop +}\frac{(1-|a_n|^2)z}{\overline{a}_n z}{\atop +\cdots}\,.$$

Then we have the characterization that

$$\lim_{n\to\infty}\int_{\mathbb{T}}\left|f-\frac{A_n}{B_n}\right|^2 dm=0$$

if and only if either f is an inner function or else $\lim_{n\to\infty} a_n = 0$; see [22, item 625, Theorem 5].

At this point it is worthwhile to point out that if f is the Schur function associated with the Carathéodory function[33] of μ (see [21, p. 25]), then, according to a marvelous and surprising theorem of Ya. L. Geronimus,

$$a_{n-1}\equiv -\overline{\Phi_n(\mu,0)}\,,\qquad n\in\mathbb{N}\,;$$

see four proofs in [21, Chapter 3] and yet another in [21, Section 4.5].

It turns out that[34]

$$(\text{weak }*)-\lim_{n\to\infty}|\varphi_n(\mu)|^2\,d\mu=dm$$

if and only if

$$\lim_{n\to\infty}\Phi_n(\mu,0)\Phi_{n+\ell}(\mu,0)=0$$

for all fixed $\ell\neq 0$; see [22, item 625, Theorem 4].

Combining these with a few more ingredients which I will not mention, we get that (weak $*$) $-\lim_{n\to\infty}|\varphi_n(\mu)|^2\,d\mu=dm$ is equivalent to $\lim_{n\to\infty} a_n=0$ unless μ is singular; see [22, 625, Corollary 2.6, p. 177].

An important application of Khrushchev's theory is the following. Given $\epsilon>0$, μ with $\log\mu'\in L^1$, and an "arbitrarily rare" gap-subset Λ of $\mathbb{N}$, there exists a *singular* measure ν such that $(\Phi_n(\mu,0))_{n=0}^{\infty}$ and $(\Phi_n(\nu,0))_{n=0}^{\infty}$ differ on Λ only and $|\Phi_n(\nu,0)|<\epsilon$ for $n\in\Lambda$; see [22, item 625, Corollary 9.2, p. 237]. This shows that the inverse problem of recovery of μ via the Schur parameters of the Schur function associated with the Carathéodory transform of μ is extremely unstable.

Another example is the theory of OPUC (and OPRL) with varying measures. Extensions of Rakhmanov's theorem and Szegő's theory to OPUC with respect to varying measures when for each n, the measure μ_n and $\varphi_n(\mu_n)$ depend on n (see [22, items 251, 252, 709, and 710] and [1]) have diverse applications. For example, they are useful in approximation of the Cauchy transform of measures supported on the real line and their meromorphic perturbation by means of interpolating rational functions (see [22, items 708 and 711]), in comparing the asymptotic behavior of two sequences of polynomials orthogonal with respect to two "related" measures supported on unbounded real intervals (see [22, item 712]), in finding ratio asymptotics of OPUC with respect to a measure μ supported on an arc γ of the unit circle such that $\mu'>0$ a.e. on γ (see [22, item 100]), in obtaining strong asymptotics when μ satisfies a Szegő–type condition of logarithmic integrability on γ (see [22, item 102]), and in ratio asymptotics of multiple orthogonal polynomials that distribute their orthogonality relations with respect to a Nikishin system of measures (google "ratio asymptotics Nikishin systems").

Results in [22, items 100, 466, 855–857, and 859] motivated the characterization of those sequences of OPUC that satisfy ratio asymptotics over a period in terms of the asymptotic periodic behavior of the absolute value of the recursion coefficients and the ratio of consecutive recursion coefficients. This led to new classes and

[33] AKA Herglotz or Herglotz–Riesz transform.

[34] For weak convergence, see, e.g., `http://mathworld.wolfram.com/WeakConvergence.html`.

examples of OPUC whose OPRL analogues had been studied by Ya. L. Geronimus and by baby boomers such as A. I. Aptekarev, J. S. Geronimo, and W. Van Assche. A good starting point is [22, Section 13.4].

Now we've arrived at the next...

Question. Why OPUC?

I spent a great deal of time trying to come up with an eloquent explanation that would convince the reader that OPUC is a subject worthy of study. In the end, I decided that no matter what I do, it will be less convincing than using Barry's own words from the introduction of [21].[35]

Although OPUC have an intrinsic interest, they also arise in many applications. While Barry's book does not focus on these applications, some appear in passing, and it is useful to describe some background.

(1) *Stationary Stochastic Processes, Filtering, and Circuit Theory.* Let $(\omega_j)_{j\in\mathbb{Z}}$ be a sequence of complex random variables with finite variance, that is, $\mathbb{E}(|\omega_j|^2) < \infty$, and suppose they are stationary, that is, the joint distribution of $\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_k}$ and of $\omega_{j_1+\ell}, \omega_{j_2+\ell}, \ldots, \omega_{j_k+\ell}$ are identical for all $\ell, j_1, \ldots, j_k$. Then, there is a measure μ[36] on $\mathbb{T}$ such that

$$\mathbb{E}(\overline{\omega}_j \omega_k) = \int e^{i(k-j)\theta}\, d\mu(\theta)\,.$$

Many ideas from OPUC are relevant in this context. From this point of view, a special case of one of Szegő's theorem determines when ω_1 is a.e. a function of $(\omega_j)_{j=-\infty}^{0}$, that is, when the past determines the future. These ideas have been used in geophysical data processing and in speech processing (google, say, "signal processing szego" or "speech processing szego").

(2) *Schur Functions and H^∞.* As Ya. L. Geronimus' theorem shows, OPUC and the theory of Schur functions are intimately related; see [21, Chapter 3]. One can take results from OPUC and translate them into results on Schur functions that make no mention of measures or OPs and vice versa. For instance, G. Baxter's theorem can be interpreted as follows. A Schur function f with Schur parameters $\gamma = (\gamma_n)_{n=0}^\infty$ and Taylor coefficients $\sigma = (f^{(n)}(0)/n!)_{n=0}^\infty$ has $\gamma \in \ell_1$ if and only if $\sigma \in \ell_1$ and $\sup_{|z|<1}|f(z)| < 1$; see [21, Section 5.2] (google, say, "schur szego").

(3) *Toeplitz Determinants and Matrices.* A Toeplitz matrix is a finite or semi–infinite matrix, T, whose matrix elements have the form $t_{nm} = c_{n-m}$ for some sequence (c_k) that is polynomially bounded in k. The distribution $s(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$ is called the *symbol* of c. OPUC are closely related to Toeplitz matrices whose symbol is $d\mu$. In fact, Szegő introduced OPUC in 1920 in his study of the asymptotics of determinants of Toeplitz matrices. There is considerable interest because of explicit applications in cases when the symbol is not a measure, for example, when it is a continuous, nonvanishing, complex-valued function with nonzero winding numbers. Toeplitz matrices arise in problems in statistical mechanics, and in discretization of some differential equations (google, say, "toeplitz szego" or "toeplitz symbol szego").

(4) *Random Matrix Theory.* There is a close connection between random matrices, Toeplitz determinants, and OPUC so that, for example, many papers on

[35] In what follows, I somewhat shortened and edited what Barry wrote.

[36] μ is not necessarily normalized to $\mu(\mathbb{T}) = 1$.

the subject discuss the Christoffel–Darboux formula. The most direct connection is to CUE, the circular unitary ensemble of random unitary matrices, but this is related to GUE, the Gaussian unitary ensemble of complex Hermitian matrices and its variants. As a matter of fact, applying Gram-Schmidt to the rows of a random Hermitian matrix yields a random unitary matrix.[37]

(5) *Spectral Theory.* Barry's interest in this subject was stimulated by OPUC as a laboratory and playground for ideas and methods in the spectral theory of Schrödinger operators. There are four families of closely related problems, namely, one-dimensional Schrödinger operators, Jacobi matrices, OPUC, and the theory of Krein systems. In many ways OPUC is technically the simplest of the four. For example, the "sum rule"[38]

$$\prod_{n=0}^{\infty}(1-|\Phi_n(\mu,0)|^2) = \exp\left(\int_0^{2\pi} \log(\mu'(\theta))\,\frac{d\theta}{2\pi}\right)$$

was written down in complete generality by S. Verblunsky in [22, item 1066] in 1935, but the analog for Jacobi matrices was only found by R. Killip and Barry in [22, item 633] in 2001. On the other hand, spectral theory has been most thoroughly developed for discrete Schrödinger operators. [22] is largely the carryover of ideas developed in the theory of Schrödinger operators to OPUC. In fact, since some of the Schrödinger theory is material that is only available in papers, [22] represents a first book on general spectral theory that covers the developments of the past 15 years.

(6) *Unitary Operators.* Every unitary operator with a cyclic vector is, by the spectral theorem, unitarily equivalent to multiplication by z on $L^2_\mu(\mathbb{T})$ for some μ. Thus OPUC is, in a sense, the theory of unitary operators. This way of thinking has not impacted the literature much.

(7) *Geophysical Scattering.* There is a model of scattering from multilayered media in which $\Phi_j(\mu,0)$ is related to the amount reflected from layer j and the product $\prod_{j=1}^{n}(1-|\Phi_j(\mu,0)|^2)^{1/2}$ is a probability of transmission through the first n layers.

(8) *A Model in Solid-State Physics.* G. Blatter and D. A. Browne had a model of conducting rings that led them to study generalized CMV[39] matrices. The connection of OPUC to this model remains to be explored.

(9) *Commutant Lifting Theorems.* D. Sarason realized that analytic functions on the unit circle with measure boundary values are a special case of what has come to be called the theory of dilation of operators. There is a large literature on this subject, as discussed in C. Foias' and A. Frazho's book [22, item 358].

(10) *Combinatorics.* There are important connections among Toeplitz determinants, OPUC, and combinatorial generating functions. The seminal paper is I. M. Gessel's [22, item 411].

(11) *OPRL.* Szegő discovered that all OPRL systems living on a finite interval can be mapped to OPUC systems; see [26, Section 11.5]. In particular, Szegő obtained asymptotics for certain OPRL using OPUC. This theme clearly increased

[37]By now you must have figured out what to google. Otherwise, please call me. I am on EST/EDT.

[38]Take logs.

[39]Regarding the name "CMV", see the FAQ at the end of this review.

the interest in the OPRL community for OPUC. For example, it was a motivation for Rakhmanov's work.

Thank you, Barry, for telling us why studying OPUC is a legitimate obsession.

Now that the connection between OPUC and OPRL has been established, the list can go on. Number theory, continued fractions, approximation, interpolation, the moment problem, summability, orthogonal series, singular integrals, quadratures, numerical analysis, random walks, and so forth.

The time has come to talk about Barry's book on OPUC.

It would not be entirely unexpected, and, in fact, it would be to some extent even natural and appropriate if the casual reader's first reaction to Barry's OPUC were "Thank G.d, the Almighty, that Barry did not set out to write a book about OPs in general and chose to limit himself to a tiny subset of the area focussing on OPUC."

For starters, it's a monstrosity in 2 volumes, 1044 + xxv + xxi pages (counting the twice partially repeated bibliography): it has 2 prefaces, 13 chapters, 4 appendices, 2 author and 2 subject indices; it lists 1,119 items in the references; it has 481 theorems,[40] 95 lemmas, 789 proofs, 3,858 numbered formulas and 1,484 non-numbered displayed formulas for a grand total of 5,342; it mentions my name 110 times and Barry's own name 173 times (not counting the references); and so forth and so forth.[41]

At this point the reader would be better off reading [24]. It really is an excellent summary of what awaits in Barry's OPUC.

Once that has been taken care of, the next step is to ask around to see how the experts would react to Barry's OPUC. This is precisely what I did, and here is a selection of the responses I received from people looking at OPs from various vantage points.[42]

Harald Widom[43] wrote:

> The strong Szegő limit theorem gives asymptotics of the Toeplitz determinant $D_n(f) = \det(\hat{f}_{j-k})_{j,\,k=0}^{n-1}$, where the $\hat{f}_k$ are the Fourier coefficients of the function f (the "symbol") defined on the unit circle. The theorem says that under appropriate conditions
>
> $$\lim_{n\to\infty} \frac{D_n(f)}{\hat{L}_0^n} = \exp\left(\sum_{k=1}^{\infty} k\,\hat{L}_k\,\hat{L}_{-k}\right)$$
>
> where $L = \log f$ and $(\hat{L}_k)$ are its Fourier coefficients.
>
> This formula was discovered by Lars Onsager[44] in his work on the Ising model. In a 1950 correspondence he stated the general result (as a conjecture) in a form equivalent to the one stated above, and he proved it for $f(z) = p(z)/q(z^{-1})$ where p and q are polynomials; cf. A. Böttcher's [22, item 137]. The conjecture came to Szegő via S. Kakutani, and in 1952 he proved it when f was a sufficiently smooth positive function in [22, item 1028]; cf. [21, pp. 331–333].

[40]Use "grep 'begin[{]theorem[}]' c*.tex | wc -l".

[41]Each number in this sentence must be preceded by "at least".

[42]I somewhat edited their responses, and they all approved the current version.

[43]Professor Widom needs no footnote.

[44]See `http://nobelprize.org/chemistry/laureates/1968/onsager-bio.html`.

In [21, Chapter 6], Simon presents five different proofs of this result, for the most part in its sharpest form. The condition is that

$$\sum_{k=-\infty}^{\infty} k\,|\hat{L}_k|^2 < \infty\,,$$

in other words that $\log f$ be in the Hölder space $H^{1/2}$. For a positive function with integrable logarithm the formula holds whether the right side is finite or infinite. There is also an extension to the case of symbols that are measures. All of these proofs are quite technical. If one is willing to forego the greatest generality, then there is a very simple operator-theoretic proof discovered independently by E. L. Basor and J. W. Helton in 1980,[45] and by A. Böttcher and B. Silbermann [22, item 141] in 1980, where the right side emerges as the determinant of a multiplicative commutator. These operator-theoretic ideas come into play in Simon's proof of a formula discovered by K. M. Case and J. S. Geronimo [22, items 395 in 1979 and 396 in 1980] and rediscovered by A. M. Borodin and A. Okounkov [22, item 135 in 2000], which has played an important role in recent advances in the study of the asymptotics of growth models and related probabilistic problems. This formula gives an exact expression for the quotient on the left side. It involves Hankel operators, which behave much more nicely than Toeplitz operators since they have discrete rather than continuous spectra, and this is why the formula has been so useful in asymptotic problems where the symbol also depends on n.

One connection between Toeplitz determinants and random matrix theory can be seen in the Bump–Diaconis proof of the strong Szegő theorem presented in [21, p. 348], which is based on the observation that a Toeplitz determinant is an integral over the unitary group $U(n)$. From other questions in random matrix theory and elsewhere, there arises the problem of finding a substitute for and/or a generalization of the asymptotic formula when the symbol does not necessarily have a logarithm belonging to $H^{1/2}$, for instance, when it has a zero, an infinity, or a jump. As a result of their work in statistical mechanics, M. E. Fisher and R. E. Hartwig [22, items 351 and 352 in 1969] made a far-reaching conjecture for a class of symbols having even several zeros or singularities of a certain kind. These are now known as Fisher-Hartwig symbols. The first significant result in this direction was obtained by A. Lenard [22, item 690 in 1972], who determined the asymptotics in the case $f(z) = |1-z|^{\alpha}\,|1+z|^{\beta}$, and found out that a factor $n^{(\alpha^2+\beta^2)/4}$ must be inserted into the denominator on the left side, and that the resulting limit is expressible in terms of the Barnes G–function,[46] and, thus, verifying the conjecture in this case. In the intervening thirty+ years the Fisher–Hartwig conjecture was taken up by many

[45] See MR0565749 on MathSciNet.

[46] See `http://en.wikipedia.org/wiki/Barnes_G-function` and `http://www.answers.com/topic/barnes-g-function`.

mathematicians, it was proved in progressively greater generality, and it is now probably in its final form.

Percy Deift[47] wrote:

> Beginning in the early 1800's, OPs were studied on an individual basis by such masters as Legendre, Jacobi, Laguerre, Chebyshev, and Hermite, amongst many others. The situation changed dramatically in 1894–95, when Stieltjes published his classic memoir [22, item 1002] on continued fractions and the moment problem, providing *en route* the foundation for a general theory of OPs. By the 1930's the subject had left its moorings in the moment problem, and had found anchorage in the following specialized Gram–Schmidt process.
>
> Let μ be a measure with finite moments on $\mathbb{C}$, and let $p_n(z) = \gamma_n z^n + \dots$, $\gamma_n > 0$, $n \geq 0$, be the orthonormal polynomials obtained by orthogonalizing $(z^n)_{n=0}^{\infty}$ with respect to μ. This is the starting point of view taken up by Szegő in 1938 in his celebrated text [26], and it remains the standard, most efficient, and most flexible entry point to the theory of OPs to this day.
>
> When μ is supported on the real line, it has long been known that the OPs satisfy a three term recurrence relation
>
> $$a_n p_{n-1}(z) + (b_n - z)p_n(z) + a_{n+1}p_{n+1}(z) = 0\,, \qquad n \geq 0\,,$$
>
> for suitable real parameters $b_n = b_n(\mu)$ and $a_n = a_n(\mu)$ where $a_n > 0$ for $n > 0$ and $a_0 \equiv 0$. In other words, for each $z \in \mathbb{C}$, $p(z) = (p_0(z), p_1(z), p_2(z), \dots)^T$ is a generalized eigenfunction for the Jacobi operator
>
> $$L = L(\mu) = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \dots \\ a_1 & b_1 & a_2 & 0 & \dots \\ 0 & a_2 & b_2 & a_3 & \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}\,,$$
>
> that is, $(L - z)p(z) = 0$. Now L is real and symmetric on $D_0 \stackrel{\text{def}}{=} \{u = (u_0, u_1, u_2, \dots)^T \in \ell_{2+} \;:\; u \text{ has compact support}\}$, and hence has self–adjoint extensions. For simplicity, let us assume that L is essentially self–adjoint with a unique self–adjoint extension $\hat{L}$, and let $d\mu = d\mu(L)$ be the spectral measure for $\hat{L}$ in the cyclic subspace generated by $e_0 = (1, 0, 0, \dots)^T$ and $\hat{L}$.
>
> Here is the basic question. If one starts with a measure μ_0 on $\mathbb{R}$, constructs the Jacobi operator $L = L(\mu_0)$, and then constructs the spectral measure $\mu(L(d\mu_0))$ for $\hat{L}$ as above, then what is the relation of $d\mu(L(\mu_0))$ to $d\mu_0$? Modulo technicalities, the answer is that $d\mu(L(\mu_0)) = d\mu_0$. In other words, the orthogonal polynomial construction $\mu_0 \mapsto L(\mu_0))$ is just the solution of the inverse problem for the spectral map $L \mapsto d\mu(L)$. This viewpoint was taken up, in particular, by N. I. Akhiezer in his beautiful monograph [22, item 17 in 1965] on OPs and the moment problem, and also by

[47] Percy is Barry's only student who is not younger than Barry.

Simon in his extensive review of OPs on the real line [22, item 974 in 1998]. The recognition of OPs as a partner in a spectral and inverse spectral problem marked a new phase in the development of OPs, and coupled OPs to the extensive and powerful developments that have taken place in the spectral theory of discrete Schrödinger operators over the last 20–30 years. Moreover, as first recognized by M. Stone, every self–adjoint operator in a separable Hilbert space is a direct sum of Jacobi operators L, and consequently the $d\mu \leftrightarrow L$ duality locates OPs, at least at the conceptual level, in a central position in analysis; see [22, item 1005 in 1932].

Finally we come to Simon's book on OPUC. Such polynomials, corresponding to measures μ that are supported on the unit circle, were first singled out for general study by Szegő, but until very recently a crucial ingredient in the theory was missing. The situation is as follows. Guided by our experience with OPs on the line, we expect that μ should now correspond to the spectral measure of some unitary operator U. Just as $L = L(\mu)$ does not contain "too many" variables, one needs to define a unitary $U = U(\mu)$ with "enough" but not "too many" variables, so that the map $\mu \mapsto U(\mu)$ is bijective. Such an operator $U = U(\mu)$ was found only very recently by M. J. Cantero, L. Moral, and L. Velázquez [22, item 181]. $U(\mu)$ is called the CMV matrix and has a pentadiagonal structure, and indeed the map $\mu \mapsto U(\mu)$ is the inverse of the spectral map $U \mapsto \mu(U)$ as before.

In his extraordinary book, Simon develops the theory of OPUC along the following lines. Part 1 focuses on the map $d\mu \mapsto U(\mu)$, reworking the classical theory with great insight and economy, and also adding in many new results. Part 2 focuses on the map $U \mapsto \mu(U)$ which Simon analyzes, as in the case of OPs on the line, with all the power of the spectral methods that have been developed over the last 20–30 years.

As a subject, OPs has demonstrated remarkable longevity and vitality. The subject continues to evolve and reach out to different parts of mathematics. In 1992, Fokas, Its, and Kitaev [4] showed that OPs could be rephrased as a Riemann–Hilbert problem. This initiated the most recent phase for OPs, and has coupled OPs to the powerful non-commutative steepest descent methods that have emerged in the last 10–15 years in the asymptotic analysis of integrable systems. But this is another story...

Alphonse P. Magnus[48] wrote:

We knew Reed & Simon's books [22, items 866–869] with new and useful views on spectral theory. Then, when there was such a rage

[48] In view of what Alphonse says, it might be difficult if not impossible to believe, but I swear to G.d it's true, that he is one of the most reserved and low-keyed persons I ever had the pleasure to be friendly with.

for fractals, when everybody wanted to draw devil's staircases, Simon serenely gave us solid knowledge on these matters, always recognizing contributions of other people; see, for instance, his paper on Kotani theory [22, item 966].

Now, after M. Stone and N. I. Akhiezer, he decided to show us the full strength of spectral theory in questions of OPs. Instead of delivering a singularly discontinuous spectrum of isolated results and remarks, he gave us an enormously documented compendium, unearthing completely forgotten contributors, often giving several proofs, including by himself, of statements, shedding spectral light on the most difficult theorems, and opening new realms.

A very unusual and energetic combination of a reference book, a copious survey, and an exposition of a lot of new findings.

Sergey Khrushchev[49] wrote:

> This is a Great Book! OPUC appeared for the first time in early papers by Szegő. Since then no book so comprehensive like this one by Simon was written. The topic waited for an author for almost 90 years. Taking a look back it is not surprising now that it was Simon who became its author. To write it, one should have a very deep understanding both in Schrödinger operators and in function theory. In 1920–30's the technique of continued fractions commonly used for OPRL was replaced with Hilbert spaces and operator theory. Weyl's limit circle-point theory is a good example, since it was an extension of the Euler–Wallis formulas to the continuous case. The motivation for this book is very well explained in the introduction. So it is a generous return to OPUC what they lost many years ago from the mentioned switch of interests. As I said, it is a comprehensive book, but it is so skillfully written that when you read it you may even forget about this important fact. The explanation lies, of course, in the expository talent of the author. But there is also another point. Dozens of proofs are revised and made very clear. Simon also contributed his own new results. Had I not seen myself how this was done, I wouldn't have believed that such a massive work could be completed in a year and a half.

Guillermo (Bill) López Lagomasino[50] wrote:

> The general theory of OPs has seen a dramatic development in the last 25 years, in particular, the theory of OPUC. To a great extent this is due to the systematic use of tools coming from different areas of mathematics. Namely, from spectral theory, potential theory, boundary value problems, and the Riemann–Hilbert problem. This led to a considerable number of new results. The need

[49]Sergey and I had been classmates in college and have been friends since 1966. Googling him can lead to unexpected results. In fact, in this millennium, within a 6-month period, we had two unrelated Sergey Khrushchevs giving lectures in the very same classroom in our math building at Ohio State (true story).

[50]Bill was a classmate of E. A. Rakhmanov and a student of A. A. Gonchar. He is a typical representative of our turbulent times: born in Havana, grew up in Cleveland, then back to Havana, then Moscow, then back to Havana, and now he is a Spanish citizen living in Madrid.

> for an updated reference book summarizing the main achievements from a general perspective had become indispensable. Some notable attempts had been made which partially covered this gap; see, for example, [22, items 721 by D. S. Lubinsky and E. B. Saff, 823 by E. M. Nikishin and V. N. Sorokin, 930 by E. B. Saff and V. Totik, and 997 by H. Stahl and V. Totik]. These books focus mainly on the authors personal experience in specific areas of the subject matter. None of them deal with the basics and foundations of the theory. Perhaps the *Lecture Notes* monographs written by W. Van Assche, D. Lubinsky, and V. Totik, and the book by D. Lubinsky and E. Levin should have also been included in the bibliography of Simon's book. The task had perhaps become too challenging because of the effort involved and the responsibility of being fair with all the authors involved.
>
> According to his own words, Simon recognizes to have become consciously exposed to the theory during the initial editorial stages of [22, item 464] published in *Comm. Math. Phys.* in 2001. Until very recently, his name has not been on the list of participants of the numerous conferences held dedicated to this subject. His first paper with the words "orthogonal polynomials" in the title appears to be [22, item 976] published in *J. Approx. Theory* in 2004. In February of 2002, Simon decided to write an article to carry over all the Schrödinger operators to OPUC. Eventually, this paper grew to this magnificent book. In doing so, Simon kept contact with several specialists in the area submitting the text to constant criticism and feedback. Personally, I found it impossible to keep up with the speed with which he produced new material and improvements. Reading the text one becomes marveled by the extent and detail to which Simon is familiar with the work and methods used by other authors. At the same time, he introduces a great number of new ideas, extensions, and simplifications. The presentation is impeccable and enlightening. For some of the most relevant results, several proofs are provided when they offer different views and insight. The material presented covers all the basics starting from the initial results up to the present state of the art of the theory of OPUC. In the second volume, the significance and links with spectral theory is displayed. I am sure that all who have contributed to some extent to the theory will feel recognized and surpassed in this magnificent book.

Franz Peherstorfer[51] wrote:

> This is not a usual book on OPUC, it is a firework of new and known ideas and facts on the theory of Carathéodory and Schur functions, on the associated probability measures, and on their link to the recurrence coefficients. In each chapter, one can recognize the wide knowledge and great experience of the author. Especially

[51] I believe that Franz studied OPs using more techniques than anyone else. His web page, `http://www.dynamics-approx.jku.at/peherstorfer.html`, lists wine as his hobby.

impressive and instructive are the explanations of the cross connections and of the influence in other fields.

For instance, just to give a taste, there are six methods of attack to the Strong Szegő Theorem. Among others, one is based on representations of the group of $n \times n$ unitary matrices, one on a combinatorial approach basically due to M. Kac, a proof using ideas and methods of statistical mechanics, and another one based on the remarkable determinant formula for Hankel operators.

Another example is the investigation of periodic recurrence coefficients by functions meromorphic on a hyperelliptic Riemann surface cut along arcs of the unit circle which involves the Theorems of Abel, Jacobi's inversion problem, and their link to completely integrable systems.

Everybody will find something new in this book which covers almost everything on OPUC including numerous historical remarks and an extensive reference list.

This book will become a standard reference work on OPUC, likely a kind of a bible as Szegő's book [26].

Francisco (Páco) Marcellán[52] wrote:

From the classical monograph of Szegő's [26] to the magnificent book by Simon, OPUC constitutes a good example of the evolution of the mathematical tools and the applications of the subject.

OPRL have long been popular within the scientific community, for instance, for their close connection with second order differential equations of mathematical physics. Very few similar examples of OPUC have been studied in detail.

Using complex analysis as the primary tool, OPUC had numerous contributions by Ya. L. Geronimus and G. Freud. Researchers interested in applications in signal theory, including N. Levinson, T. Kailath, and Y. Genin, stimulated the interaction with other mathematical disciplines such as numerical linear algebra.

In the 1980's, techniques based on potential and approximation theory focused the attention of many experts of the baby boom generation. In the 1990's the analysis of spectral properties of the Hessenberg matrix opened a new perspective that led to the very recent CMV representation successfully explored by Simon and coworkers in the last five years. In particular, the rediscovery of the Verblunsky papers [22, items 1066–1070] by Simon has changed the standard terminology used for the recurrence coefficients. The latter play a key role in the modern theory of OPUC. New proofs, new ideas for future work, an integrated approach of many areas of mathematics and mathematical physics as well as an impressive updated set of up to date references give an added value to Simon's book which, in my opinion, will become one of the basic references in the theory of OPUC in the 21st century.

[52]Páco created the Spanish school of OPs, and he helped me to qualify for the Boston Marathon, which I ran in 2000.

I would have loved to see a short discussion of the Riemann–Hilbert approach for the asymptotics of OPUC which is briefly mentioned in [21, p. 332 and p. 403].

Doron S. Lubinsky[53] wrote:

Simon's entry into OPUC has revolutionized OPs and his book is ample testimony of that. Amongst the major achievements is the realization that there are higher order analogues of Szegő's theory. Szegő showed that $\log \mu' \in L^1$ iff $(\varphi_n(\mu, 0))_{n=0}^{\infty} \in \ell^2$. Simon and his collaborators have obtained, and continue to obtain, higher order analogues. This dramatic breakthrough is presented in the book with numerous proofs.

The book is remarkable for its myriad of connections, and for its multiple proofs. It is remarkable for being so up to date, and for containing so much original material. Never before have operator theory, matrix theory, complex function theory all been so intertwined with OPUC as they are now. Never before has there been such a complete and accessible presentation of the strong Szegő limit theorems.

This book is an event that will change the way orthogonal polynomials are studied and applied. It is the Szegő [26] of the 21st century.

Peter Yuditskii wrote:

It was really amazing to watch the speed with which a hypothesis became a theorem during the writing of Simons' book, and thus creating space for a new conjecture or problem! Probably it is not surprising that the central idea of the unity of OPUC and OPRL problems with spectral problems for differential operators came, in part, from Kharkov (via Leonid Golinskii) where this idea has been pervasive due to N. I. Akhiezer, M. G. Krein, V. P. Potapov, and V. A. Marchenko. Professionally, I was especially interested in the discussion of sum rules and almost periodic CMV matrices. Together with Franz Peherstorfer and Sasha Volberg, we were also trying to apply Case's identity to a description of the "Szegő function over B–product" spectral class of Jacobi matrices. However we absolutely overlooked a brilliant idea by Rowan Killip and Simon to use these identities for a complete spectral characterization of the Hilbert–Schmidt class perturbations.

I remember very well the eagerness with which we, students at the University of Kharkov, were waiting for each new volume of Reed & Simon's books [22, items 866–869]. This was not only because of witty epigraphs that we quoted to our girlfriends and/or boyfriends elevating the level of our own cleverness in their mind. The current generation of students will definitely share our joy going through the new book of Simon.

[53] Doron claims that he was my postdoc, but, in fact, it was the other way around.

Hrushikesh N. Mhaskar[54] wrote:

> Simon's book is the first book fully devoted to the theory of OPUC since Geronimus' [22, item 407]. It is an encyclopedic treatment of the question of determining the properties of OPUC, including their underlying measure, zeros, and so forth, given the sequence of recurrence coefficients. This is not a mainstream question in the theory of OPRL. Therefore, I am surprised to see so much research devoted to this question in the context of OPUC. I missed some results, which I would have expected, such as N.'s asymptotics of the derivatives of OPUC, the K. Pan & E. B. Saff theory of the behavior of the zeros near the singularities of the Szegő function, and the work of V. Andrievskii, H.-P. Blatt, and myself on local discrepancy theorems and Pollaczek polynomials. Of course, this is not surprising in such a monumental effort, and does not diminish its value in any way.
>
> The book is very well written, includes many many new ideas, focused around the proofs of a few important theorems in the theory. The historical comments and the strong opinions of the author on the credits make for a lively reading. Although the book is not suitable as a textbook, a good course can be based on selections from the book. I expect the book to motivate a lot of research in the area.

Serguei (Sergey) Denissov[55] wrote:

> In the variety of different one-dimensional orthogonal systems, OPUC play a special role. Analytically, they are somewhat easier to study and often serve as a toy model for different other cases, e.g., bounded Jacobi matrices, Krein systems, Dirac, and Schrödinger operators. Thus the subject not only allows us to develop a more or less complete theory, but it also serves as a background for many other fields in mathematics such as classical complex analysis, mathematical physics, probability, representation theory, and even partial differential equations.
>
> In his book, Simon gives an important new look at the theory of OPUC, making it far more complete than it has ever been. In the first volume, he develops the classical theory. Among the different new results and methods given, I want to emphasize the extensive use of the matrix representation for the shift operator; see the CMV matrix in [21, Chapter 4]. That representation made it possible to use different techniques of spectral theory and it is absolutely crucial for the understanding of the subject. In the second volume, various ideas in mathematical physics (spectral theory of Schrödinger operators) are applied to the OPUC; see [22, Chapters 10 & 12]. The truth is that people working in approximation theory and in mathematical physics did not always realize the deep link between these two fields. In my opinion, this book fills the gap,

[54] Hrushikesh was my first Ph.D. student, but only because his original advisor, G. Freud, died in 1979, just when Hrushikesh was about to finish his dissertation.

[55] Sergey was Barry's postdoc. Occasionally his last name is spelled as Denisov, one 's'.

> at least for discrete orthogonal systems. It also suggests some new interesting open problems to consider; see [22, Appendix D].
>
> Simon largely focuses on recent results in the field. He does not discuss extensively some analytical tools, say, the potential theory, but for a good reason. The book is already more than a thousand pages and there are already great books on that subject; see, e.g., [22, item 930]. The exposition is lucid and it is done with much care in a style that makes the book accessible to almost anyone. In short, we now have a big piece of work that does, in my opinion, considerably push forward our knowledge of the subject.

I think of Barry, and some even might agree with me, as one of the most opinionated persons there is,[56] and his essay "Twelve Great Papers" [22, p. 975] is yet another proof of that. I should feel honored for having my name mentioned in it, but, for G.d's sake, where is Akhiezer, Kolmogorov, Krein, Levinson, Smirnov, and Totik,[57] just to name a few? In addition, what about 1977 through 2001? No great papers during the most productive years? Knowing that Barry will strike back and destroy my arguments point by point anyway, I'd rather shut up.[58]

I know, I know, the reader is eagerly awaiting more criticism. However, I don't think I have the courage to say anything publicly that might haunt me for the rest of my life. For instance, when I noticed that the bibliography starts with item [6] on [21, p. 425], Barry immediately pointed out to me that there is an explanation for it; see [21, p. xiv, line 23]. Professor Simon, please give me a break.

OK, just one intzy-wintzy thing. Barry dismisses V. A. Steklov's conjecture with two brief historical remarks; see [21, middle of both p. 121 and p. 134]. In short, Steklov expected that (lower) bounds on the density of the measure μ alone can lead to boundedness of OPRL (and OPUC). I am ready to admit that Steklov's conjecture is by no means as deep and as quintessential as N. N. Luzin's[59] conjecture, which led to Carleson's theorem.[60] In addition, Steklov's conjecture turned out to be false. Still, I would have loved to see actual theorems and proofs, especially since they would have given an insight to constructing incredibly complicated counter-examples in OPs. In addition to the papers by Rakhmanov [22, items 886, 887, and 888], the reader may want to consult papers by M. U. Ambroladze.[61]

Although OPUC might not be bounded in general, it turns out that $\log \mu' \in L^1$ implies the $(\mathrm{C},1)$-boundedness, at least almost everywhere. This fact, proved in [13], is dismissed by Barry, unless one counts a cryptic remark in [21, formula (2.2.107), p. 134]. In addition, the $(\mathrm{C},1)$-limit, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} |\varphi_k(\mu, z)|^2 = \frac{1}{\mu'(\theta)}\,, \qquad z = \exp(i\theta),$$

[56] I wonder if this is why he has not been elected to NAS yet.

[57] Totik is mentioned as a kind of a backup player on page 976.

[58] Barry could have added his own book to the list to create an infinite recursion.

[59] Let me plug here [14, Extras/Luzin.html], which is yet another proof of how wicked the Soviet system was and how so many distinguished Soviet mathematicians failed some elementary tests of human decency. *Golden Years of Moscow Mathematics*, published by the AMS, is also an excellent source of information on Luzin.

[60] See, e.g., `http://eom.springer.de/L/l061070.htm`.

[61] See MR1110069 and MR1019042, or if you don't have access to MathSciNet, then try Googling "MU Ambroladze". He is the brother of the one mentioned on [21, p. 24], and he is no longer pursuing math.

which holds almost everywhere, has been proved under rather weak conditions, say, $\log \mu' \in L^1$ in [22, item 761]. Barry calls this condition "more restrictive," which is technically correct but misleading. I admit that the proof of "$\ldots = \ldots$" above is rather technical, so that Barry was correct in not subjecting the reader to it. However, "$\liminf \ldots \geq \ldots$" and "$\limsup \ldots \leq \text{const} \ \times \ldots$" are not at all that complicated, and some of the self-contained arguments from [13] could have been reproduced; see [29] for a most general result.[62]

Now, if every person who (i) has ever contributed to OPUC, (ii) is still amongst us, and (iii) reads this paragraph prior to its publication had the right to submit a list of omissions, then we would end up with a list of biblical proportions. What a nightmarish prospect.

I hate to admit it, but Barry is the most organized and the most well–prepared person I have ever met, and this shines through the book as well. It is probably the most extensively cross–referenced traditionally published book in the history of mathematics publishing. The person responsible for this, Cherie Galvez, deserves the gratitude of all who will ever take advantage of the meticulous attention to organizational details. There is no way to even try to remotely illustrate this[63] in a review such as this one. Instead, the reader had better run to the nearest library and take a look for herself.[64] There is no other book I have ever seen that can be compared to Barry's in this respect—not Szegő, not Dunford–Schwartz, not Hille–Philips, not Zygmund, not even Pólya–Szegő.

1. FAQ WRT [21, 22, 23]

QUESTION. What is [23]?

ANSWER. It's the website for [21, 22], a depository of future plans, addenda, and corrigenda. Naturally, it is "under construction".

QUESTION. Are there any typos in [21, 22]?

ANSWER. Yes. At least three. "Marchelan" in [22, item 740] should be "Marcellán",[65] and "othogonal" and "respeect" in [22, item 740] should be something else.[66]

QUESTION. Are there any minor inaccuracies in [21, 22]?

ANSWER. Yes. At least two. Barry refers to Szegő [22, item 1017] twice, and both times his comments are wrong.

QUESTION. What was one of Barry's first actions upon formally joining the OPUC community?

ANSWER. He rediscovered Verblunsky's papers [22, items 1066–1070] and set the historical record straight by renaming the recurrence and/or reflection and/or Geronimus and/or Szegő and/or Schur coefficients to Verblunsky coefficients. The new terminology seems to have been universally accepted by the OPUC community. Googling "verblunsky 'orthogonal polynomials' 'unit circle'" yielded 550+ hits in May of 2006. Interestingly, the same query with "verblunsky" omitted gave 300+K hits.

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[62] Dear G.d, please make Barry forgive me.

[63] According to the latest fashion, split infinitives are no longer pariahs.

[64] As of May 2006, in the U.S., the book costs \$149 + tax (\$119 for AMS members), which is a bargain for those who can easily afford to spend about a dime per page.

[65] Barry will find an excellent excuse that it was published in Russian.

[66] This explains why Barry "gets no respeect"; see [24, item 108].

QUESTION. Are there any major historical screwups in [21, 22]?

ANSWER. Yes. At least one, repeated at least 185 times. The entire CMV–naming business was botched. A year after his books appeared, Barry learned in an e-mail from David S. Watkins, which he forwarded to some of us in the OPUC community, that [31] has all the major results of Cantero–Moral–Velázquez's [22, item 181].

It became clear afterwards that the pentadiagonal representation for real orthogonal matrices first appeared in [25, item 5] by G. S. Ammar, W. B. Gragg, and L. Reichel in 1986 and that they found the right complex analog of the Householder algorithm in [25, item 6] in 1988. They could have plugged this into their earlier arguments to get the general pentadiagonal representation, but they didn't do the latter explicitly. For details, see, where else, Barry's paper [25].

Given his introduction of the terminology "Verblunsky coefficient", it seems puzzling that Barry insists that the name "CMV matrix" stick, so I asked him to explain the reasons. He wrote:

> Short snappy names are clearly useful but they are unfair so often that V. I. Arnold has even stated what is known as the *The Arnold Principle* that "if a notion bears a personal name, then this name is not the name of the discoverer"; see `http://pauli.uni-muenster.de/~munsteg/arnold.html`. The issue comes up often, whether a name should be changed when earlier work is unearthed. In the three years I had been pushing the name, it was used frequently enough in papers that I decided to leave it. The analogy with "Verblunsky coefficient" is not valid since, if there had been a common accepted name for them, I would never have suggested changing it but there were five different names, none used in the majority of papers, so proposing a new standardized name made sense.

QUESTION. Are there any false statements in [21, 22]?

ANSWER. Yes. At least one. According to Barry, [22, formula (13.3.15), p. 891] has 5 errors. I take his word for it. I confess, I couldn't find a single error in it, even after staring at the formula for half an hour or so.[67]

QUESTION. Are there any unfair accusations in [21, 22]?

ANSWER. Yes. At least one. In addition to Szegő, Barry also blames Freud and Máté–N.–Totik for the generally used definition of the Szegő function (see [21, Remark #1, p. 144]), whereas all of us, just like everyone else, followed Szegő so that Szegő is the guilty party all by himself. Anyway, my guess is that Barry would have liked to use the reciprocal of the Szegő function as the definition, because then some of the asymptotics could have been typeset without using fractions.

QUESTION. Is there a chance that [21, 22] will replace the Torah?

ANSWER. Not a chance, at least according to `http://en.wikipedia.org/wiki/Bible_code` and Barry Simon's `http://wopr.com/biblecodes`.

QUESTION. How many persons mentioned in [21, 22] were born in Budapest?

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[67]Considering that there are at least 5,342 displayed formulas, a scary thought crossed my mind.

ANSWER. At least 11, enough for a minyan with a backup. Can you name them? Don't cheat.[68] Can you name more?[69]

Summary. In my not necessarily humble opinion, Barry's OPUC is one of the best things that has ever happened to OPUC. It is going to keep a couple of generations of researchers on their toes. I predict that it will equal Szegő's book [26] both in terms of scientific impact and commercial success. Of course, as we all know, prediction is very difficult, especially about the future,[70] unless $\log \mu'$ is not integrable [22, item 642].

Acknowledgment. I thank all who contributed to this piece.

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[68] Erdélyi, Erdős, Freud, Lax, Nevai, Pintér, Pólya, Szabados, Turán, von Neumann, and Wigner, but not Fejér (Pécs), Fekete (Zenta/Senta), Koranyi (Szeged), Lenard (Balmazújváros), Máté (Szeged), Riesz (both in Győr), Szegő (Kunhegyes), Szőkefalvi-Nagy (Kolozsvár/Cluj), and Totik (Mosonmagyaróvár). On the other hand, Hershel Farkas (Brooklyn, New York), Fritz Gesztesy (Leibnitz, Austria), and Jim Rovnyak (Ford City, Pennsylvania) are not even Hungarian.

[69] If you like puzzles, see [17].

[70] According to Harald August Bohr's older brother, whose fourth son was awarded a Nobel Prize in physics; see `http://www.quotationspage.com/quote/26159.html` and `http://nobelprize.org/physics/laureates/1975 /bohr-autobio.html`.

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