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Rings, modules, and the total, by Friedrich Kasch and Adolf Mader, Frontiers in Mathematics, Birkhäuser-Verlag, Basel, 2004, x+136 pp., US$49.95, ISBN 3-7643-7125-0

Equivalence and duality for module categories, with tilting and cotilting for rings, by Robert R. Colby and Kent R. Fuller, Cambridge Tracts in Mathematics, Cambridge University Press, vol. 161, Cambridge, 2004, x+152 pp., US$60.00, ISBN 0-521-83821-5

To discuss these two books, I will start with a short historical review, talking about each book approximately at the point when the basic contents and approach but not all the details were available for the material of the book.

In this review, I adopt the current standard disclaimer that all rings have identity and all modules are unital. For a ring R, M as an R-module will be denoted by writing R as a subscript of M on the side on which it operates. Thus M_R will mean M is a right R-module. M_R as a submodule of K_R will be denoted $M_R \leq K_R$. Default phraseology will have the operations of a ring on a module on the right, and the symmetrical statements for left modules are assumed.

The study of general as opposed to specific rings is a product of the 20^{th} century. There was a dramatic change in the investigations pursued in the middle of that century. Prior to approximately 1950, a ring was usually defined as a set R with two associative operations, addition and multiplication, with axioms stating that R is an abelian group under addition and multiplication distributes over addition. To distinguish between this concept and the current one, such a structure has been called an rng (for ring without (i)dentity). Investigations concerned with rngs did not look at modules, although they did look at one- and two-sided ideals, and occasionally they did look at abelian groups with operators, which is what we now call unital modules. Many of the questions asked were about elements in rngs: for example, whether certain conditions forced commutativity or whether appropriate conditions forced certain elements or ideals to be nilpotent (have some power = 0) or when other conditions forced the existence of idempotents $e = e^2$ to exist. Other questions asked concerned one-sided or two-sided ideals, frequently including chain conditions on these ideals. A third set of questions concerned radicals of rngs. A radical is a 'function' σ from the class of rngs to itself (here you *cannot* mandate an identity) such that $\sigma(R)$ is a two-sided ideal of the rng R and $\sigma(R/\sigma(R)) = 0$. An example was $\mathfrak{N}(R)$ = the sum of all the nilpotent ideals. Another example was $J_r(R)$ = the intersection of all the annihilators of all quotients R/I where I is a maximal right ideal not containing R^2.

When these kinds of investigations converged, significant results often arose. For example, $J_r(R)$ was shown by Jacobson in 1945 [Jac45] to be the sum of all the right ideals $I_R \leq R_R$ of R such that for all $z \in I$ there was a z' in R such that $z + z' - zz' = 0$. Jacobson used this to show that $J_r(R) = J_l(R)$, which is the version of $J_r(R)$ obtained by replacing right by left. This ideal, often denoted $J(R)$, appears in many ring and module theoretic investigations and is called the

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Jacobson radical, or just the radical, of R. Radicals of rings have ceased to be as significant an area of research, but $J(R)$ lives on and its original definition has been extended to modules by

$$J(M_R) = \bigcap \{K_R : K_R \text{ is a maximal submodule of } M_R\}.$$

The commutative rings of algebraic geometry and algebraic number theory have ascending chain condition (acc) on (right) ideals; i.e. any chain $I_1 < I_2 < \cdots < I_n < \cdots$ of (right) ideals is finite. Rings with acc are called noetherian. Commutative noetherian rings and their finitely generated modules were studied before the midcentury revolution in general ring theory, but the acc did not have comparable influence in the study of not necessarily commutative rings. Descending chain condition (dcc) on one-sided ideals was a different matter because finite dimensional algebras over a field, for example group algebras, have the dcc.

Rings with dcc on right ideals have the property that any right ideal which is not nilpotent contains an idempotent. This implies that $J(R) = \mathfrak{N}(R)$ is nilpotent. The Wedderburn-Artin theorem showed that in the case of dcc $R/J(R)$ is a finite direct product of (finite) square matrices over division rings. A ring with dcc on right ideals is called right artinian. If R has dcc and $J(R) = 0$, the ring is called semisimple artinian.

This appears to be the appropriate place to talk about the concept of the total, which is the major theme of the book by Kasch and Mader. This concept needs some explanation, even for many ring and module theorists. Kasch and Mader define the total of a ring R, $\mathrm{Tot}(R)$, as the set of all elements $f \in R$ such that the right ideal fR does not contain any nonzero idempotent $e = e^2$. In 1948–1950, Azumaya, in [Azu48] and [Azu50], wrote about exactly this concept, proposing it as a new definition of the radical of a ring in case there was a maximum idempotent free ideal. Azumaya used the language of abelian groups with operators Ω, and just as in Kasch and Mader, after initial material on idempotents and other generalities, Azumaya then looked at the structure of the endomorphism rings of direct sums of indecomposable Ω-groups M_i such that a sum of two non-automorphisms is a non-automorphism. In *Math Reviews*, the reviewer, I. Kaplansky, inserted a remark that "the radical proposed here coincides with the Jacobson radical under various weak chain conditions ... but it seems to be quite unsuited to applications such as Banach algebras. For example, the ring of continuous real functions on the unit interval has no Azumaya radical." In fact, one can be even more specific than Kaplansky. If R has no idempotents other than 0 and 1, the Azumaya radical exists if and only if R is local. Forty years later, it appears that Kasch rediscovered this concept, and the current book brings modern language to the material that was in Azumaya's papers and adds a chapter on the total of the endomorphism ring of direct sums of torsion free rank 1 abelian groups. 'Azumaya's radical exists' is replaced by 'the total is additive'. The book is accessible to readers who have the algebraic maturity usually acquired in a beginning graduate level abstract algebra course. There are modern definitions and mathematical results quoted and mostly proved throughout the book that are appropriate for a book at this level, but everything is focused on the total rather than why these definitions and results are significant in 21^{st} century ring and module theory. Whether the results on the total of the endomorphism ring of completely decomposable abelian groups are really significant to abelian group theorists I will leave to them to decide.

Let me now return to the historical development of ring and module theory today and list some threads which dramatically changed research in the area in the middle of the 20^{th} century.

If G is a finite group and K a field, then the group algebra $K[G]$ is semisimple artinian if and only if $\mathrm{char}(K)$ does not divide the order of G. In all cases, $A = K[G]$ has the property that A_A is A-isomorphic to the vector space dual $({}_AA)^*$ of ${}_AA$. Algebras with this property are called Frobenius algebras, and vector space duality gives a great deal of information about them. In a 1939 paper, [Nak39], Nakayama called an algebra A quasi-Frobenius if A_A and $({}_AA)^*$ had the same indecomposable summands, perhaps with different multiplicities. He also showed that A is a quasi-Frobenius algebra if and only if every right ideal and every left ideal is its own double annihilator in the ring A. That is, if $X \subseteq A$ let $l(X)$ denote the left ideal $\{a \in A : aX = 0\}$, and $r(X) = \{a \in A : Xa = 0\}$; then A is quasi-Frobenius iff for all $I_A \leq A_A$ and ${}_AJ \leq_A A$, $rl(I) = I$ and $lr(J) = J$. Thrall in 1948 [Thr48] defined some generalizations of quasi-Frobenius algebras using variants of annihilator conditions similar to these. Of course annihilator conditions do not require A to be an algebra over a field, and in a 1951 paper by Ikeda [Ike52] the double annihilator conditions on both sides plus dcc were used to define quasi-Frobenius rings, to show that they had an identity, and to generalize the concept in a way similar to Thrall.

Another early paper having great impact on ring and module theory was [EM45], in which Eilenberg and Mac Lane defined the concepts of categories, functors, and natural equivalences which set up the language of homological algebra. The 1945 Eilenberg-Mac Lane concept of categories and functors was developed in the book *Homological Algebra* by Cartan and Eilenberg [CE56], which circulated in manuscript before its actual publication in 1956. The homological, categorical approach to ring and module theory required rings to have an identity. A large percentage of the basic definitions of properties of rings and modules and their homomorphisms now are phrased in categorical terms, that is, in terms of homomorphisms and the underlying structures of homological algebra such as the representable functors $\hom_R(M_R, \_)$ and $\hom_R(\_, M_R)$ and the related adjoint $M_R \otimes_R \_$ or $\_ \otimes_R {}_RM$. For example, if M is an R-module, we define: M_R is injective iff $\hom_R(\_, M)$ is exact; M is projective iff $\hom(M, \_)$ is exact; M is a generator iff every module is isomorphic to a quotient of a direct sum of copies of M; M is a cogenerator iff every module is isomorphic to a submodule of a direct product of copies of M. These definitions come in dual pairs, the second member of each pair reversing the direction of all homomorphisms in the first.

In 1940 Baer wrote a paper [Bae40] generalizing the concept of divisible groups. Without using the language, Baer proved some basic results on injective modules. An indication of the significance of the homological revolution was that the Baer paper was essentially ignored until the power of the categorical approach became apparent. Over a decade after Baer, in 1953, Eckmann and Schopf [ES53] used the new homological techniques just coming in to the study of modules and bimodules over rings to give a categorical proof of Baer's embedding of modules in injectives. They showed also there was a unique minimal (up to isomorphism) injective hull of a module which was an essential extension of the module. These results are essential to much of the work done in module theory today.

At the same time, there was a flowering of work in module theory and characterizing rings by properties of their modules. Perfect rings (every module has a projective cover) and semiperfect rings (finitely generated modules have projective covers) were introduced by Bass; Chase characterized rings such that every product of flat right modules is flat as the right coherent rings; Matlis showed a ring is right noetherian if and only if every direct sum of injective right modules is injective. Faith and Utumi showed that every injective is projective iff the ring is quasi-Frobenius, and Faith and Walker showed that every projective is injective iff the ring is quasi-Frobenius. Many of these basic results are now available in a textbook by Lam [Lam99]. Homological properties were restricted to specific modules; for example, a module is quasi injective if any map from a submodule $N \leq M$ lifts to a map from M to M, or in the language of homological algebra, $\mathrm{Ext}^1_R(N\ ,\ M) = 0$ for all $N_R \leq M_R$. Researchers such as Wisbauer [Wis91] studied the category of $\sigma[M]$ of all modules generated by a module M. Representation theory of algebraic objects, noncommutative noetherian ring theory, noncommutative localizations, homological properties of families of rings, and many other areas, such as enveloping algebras over nonassociative structures, blossomed. Unfortunately, it is not possible to discuss this more here because of space limitations.

With the new emphasis on studying categorical properties of modules, it was natural to ask when the categories of modules over two possibly different rings are equivalent in the language of [EM45] and [CE56]. With vector space duality it was also natural to ask about duality of module categories, that is, contravariant equivalences between a subcategory of the category of right modules over a ring R and a subcategory of left modules over a ring S. There had to be a change of sides here, one ring on the right and the other on the left, and the subcategories could not contain any infinite direct sums of isomorphic nonzero modules because those could not be isomorphic to their double duals. The answer appeared in a paper by Morita in 1958 [Mor58]. An equivalence between the categories of right R-modules and right S-modules is given by $M_R \mapsto [\hom_R(P\ ,\ M)]_S$ and $N_S \mapsto [N \otimes_S P]_R$ where ${}_SP_R$ is a bimodule which is a progenerator (finitely generated projective generator) as both a left S-module and a right R-module. A classical example of such an equivalence occurs when R is a field, S is $n \times n$ matrices over R, and P is $n \times 1$ matrices over R. A duality between the full subcategories of right R-modules and left S-modules containing all finitely generated modules and their submodules was given by $M_R \mapsto_S [\hom_R(M\ ,\ U_R)]$ and ${}_SN \mapsto [\hom_S(N\ ,\ {}_SU)]_R$ where ${}_SU_R$ is a bimodule which is an injective cogenerator as both a left S-module and a right R-module. The rings R with dcc and $\hom_R(\ \_\ , R)$ giving a duality between finitely generated right R-modules and finitely generated left R-modules were precisely the quasi-Frobenius rings of [Ike52]. It was recognized very shortly that this was a very important paper, and Hyman Bass wrote up an exposition with three proofs of Morita's equivalence theorem in 1962 University of Oregon lecture notes. These notes were circulated within the ring theory community and cited by other authors. One of Bass' proofs established an approach called a Morita context for discussing Morita's category equivalence theorem. A Morita context is a collection $[R, S,\ {}_RV_S,\ {}_SW_R, \theta, \psi]$ where R and S are rings, ${}_RV_S$ and ${}_SW_R$ bimodules, $\theta \in \hom(\ {}_R(V \otimes_S W)_R\ ,\ {}_RR_R\ )$ and $\psi \in \hom({}_S(W \otimes_R V)_S\ ,\ {}_SS_S)$. In particular, a Morita context is what is required to give a multiplication which turns the set of matrices $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ into an associative ring. If θ and ψ are both onto, then $\_\otimes_S W$ and $\_\otimes_R V$ are inverse category equivalences between the categories of right R modules and right S modules. This terminology is still used by some authors, but it does not address Morita duality. Other algebraists worked on the duality portion of Morita's paper. Morita equivalences and dualities and generalizations appeared in all sorts of areas related to rings and modules.

One more area of investigation which developed in the second half of the 20^{th} century is torsion theories, generalizing torsionfree abelian groups and torsion subgroups. A torsion theory on an abelian category $\mathcal{C}$ is a pair $[\mathcal{T}\ ,\ \mathcal{F}]$ of full subcategories of $\mathcal{C}$ such that (1) for every $X \in \mathcal{C}$, there is a subobject F of X such that $F \in \mathcal{F}$ and $X/F \in \mathcal{T}$, (2) $\mathcal{T}$ is the class of all $T \in \mathcal{C}$ with $\hom_{\mathcal{C}}(T, F) = 0$ for all $F \in \mathcal{F}$, and (3) $\mathcal{F}$ is the class of all $F \in \mathcal{C}$ with $\hom_{\mathcal{C}}(T, F) = 0$.

We now come to the Colby-Fuller book. It is a reference work for researchers or advanced students on Morita theory, related torsion theories, and working in subcategories of module categories much as discussed above. The authors have contributed much to the study of tilting and cotilting modules and are ideal expositors for such a book. The book presumes knowledge of basic homological algebra, including the derived functors Ext^n and Tor_n, so this requires advanced graduate level studies from the reader. The authors provide a unified approach via representable functors to equivalence and duality between subcategories of modules. Their approach is more general in nature than Bass' Morita contexts and fits more neatly into the discussion of duality. It also rather naturally includes tilting and cotilting modules. The tilting modules of the title are finitely generated modules V_R such that $\mathrm{Ext}^1(V\ ,\ W) = 0$ for all $W \in \mathrm{Gen}(V)$, where $\mathrm{Gen}(V)$ equals the class of all quotients of direct sums of copies of V. If $V = R_R$, this is just a definition of a progenerator, and all the Morita equivalence theory goes through with the original argument restricted to $\mathrm{Gen}(V)$. Similar definitions are used to generalize the other concepts occurring in the Morita theorems. Tilting modules generate torsion theories and equivalences between the categories of torsion modules and of torsionfree modules over R and $\mathrm{End}_R(V)$. They also arise in looking at highest weight modules over semisimple algebraic groups, finitely generated localizing systems of ideals in Prüfer domains, graded Lie superalgebras, representations of Artin algebras, and other areas closely related to ring and module theory. The book provides tools and some examples but is not intended to discuss the many applications in algebra.

Cotilting modules are a dual version of tilting modules and are used to study dualities. Just as in the case of Morita equivalence and duality, their study is somewhat different from the study of tilting modules because the category of modules over a ring is not its own dual. They do not have the kinds of applications tilting modules have, so they do not generate as much research. However, the concept still can be a valuable tool for researchers in ring theory and related areas, and the similarities with the discussion of tilting modules make discussion of cotilting very natural in the book. If you have the necessary homological algebra background and wish to work in homological questions in ring theory or related areas, this is a good book to have available.

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