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PARTICLE TRAJECTORIES IN SOLITARY WATER WAVES

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Abstract. Analyzing a free boundary problem for harmonic functions in an infinite planar domain, we prove that in a solitary water wave each particle is transported in the wave direction but slower than the wave speed. As the solitary wave propagates, all particles located ahead of the wave crest are lifted, while those behind it experience a downward motion, with the particle trajectory having asymptotically the same height above the flat bed.

1. Introduction

A solitary wave is a localized steady two-dimensional gravity wave of elevation propagating at the surface of water over a flat bed. That is, the only restoring force acting on the water is gravity (surface tension being neglected, since its effects are relevant only for water waves of very small amplitude), the motion is identical in any direction parallel to the crest line, and the surface disturbance is a single hump decaying rapidly away from the crest and propagating at constant speed without change of form. Originally discovered by Scott Russell in 1844 whilst conducting experiments in canals, the solitary waves are essential in our understanding of the dynamics of water waves. Since the linear theory of waves of small amplitude fails to yield any approximation to solitary waves (see [13]), the first systematic procedure to study these wave patterns was via nonlinear approximations to the governing equations for water waves in the limiting case of long wave lengths (in the shallow water regime). The first attempts by Boussinesq and Lord Rayleigh to put the experimental studies performed by Scott Russell on a firm theoretical basis culminated with the derivation in 1895 by Korteweg and de Vries of a model equation whose solitary wave solutions captured to a good extent the essential features of solitary water waves; cf. [13]. The solitary wave solutions of the Korteweg-de Vries equation were later (in the late 1960's) found to possess remarkable properties: their speed is proportional to their amplitude, and when a large solitary wave catches up to a small one, it virtually passes through, recovering its original shape and speed, the only hallmark of the nonlinear interaction being a slight phase shift. Such solitary waves are termed solitons, and their fascinating theory has become important in various branches of mathematics and theoretical physics (see [9]). On the other hand, the rigorous theory of solitary water waves started with the work of Friedrichs and Hyers [11] and was subsequently readdressed by Beale [4]. Further substantial developments of the analytical theory are mainly due to the work

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of Toland and collaborators (see [1, 2, 3, 12]) and to the work of Craig and collaborators (see [6, 7, 8], unveiling to a large extent the structure of these waves. The aim of this contribution is to describe the particle trajectories in the fluid as the solitary wave propagates on the water's free surface. The paper is organized as follows. In Section 2 we present the mathematical formulation for the solitary wave problem, and we derive some useful properties of the underlying water flow. Section 3 is devoted to the description of the particle trajectories, while Section 4 contains some comments on related problems within water wave theory - e.g. a comparison with the recently elucidated features (see [5]) of the particle motion in periodic steady water waves (Stokes waves).

2. Preliminaries

Consider a two-dimensional solitary wave of elevation and of permanent form propagating with speed $c > 0$ on the surface of water over a flat bed; see Figure 1. Choose Cartesian coordinates (x, y) so that the horizontal x-axis is in the direction of wave propagation, the y-axis points upwards, with the origin on the flat bed. Let $y = \eta(x - ct)$ be the free surface and let $(u(x - ct, y), v(x - ct, y))$ be the velocity field, the space-time dependence of the free surface and of the velocity field of the form $(x - ct)$ expressing mathematically the fact that we investigate a wave of permanent form moving with constant wave speed c.

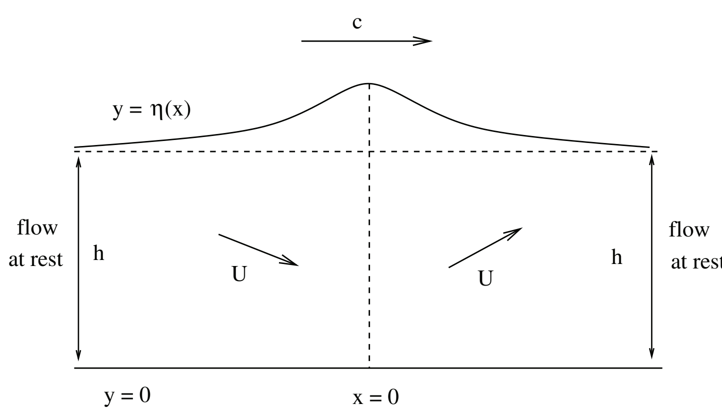

Figure 1. Solitary wave (at time $t = 0$) with asymptotic height $h > 0$, moving at speed $c > 0$. The velocity at a point in the fluid is $U = (u, v)$.

The fluid motion results as a balance between the restoring gravity force and the inertia of the system, so that within the fluid we have the equation of mass conservation

$$u_x + v_y = 0 \tag{1}$$

and Euler's equation

$$\begin{cases} (u-c)u_x + vu_y = -P_x, \\ (u-c)v_x + vv_y = -P_y - g, \end{cases} \tag{2}$$

where $P(x-ct,y)$ denotes the pressure and g is the gravitational constant of acceleration. The equations (1)-(2) reflect the fact that within this setting it is appropriate to regard water as an inviscid homogeneous fluid [13]. On the free boundary the dynamic boundary condition

$$P = P_0 \quad \text{on} \quad y = \eta(x-ct), \tag{3}$$

P_0 being the constant atmospheric pressure, decouples the motion of the air from that of the water. The kinematic boundary conditions

$$v = (u-c)\eta_x \quad \text{on} \quad y = \eta(x-ct) \tag{4}$$

and

$$v = 0 \quad \text{on} \quad y = 0 \tag{5}$$

express the fact that particles do not leave the free surface, respectively the fact that the rigid bed is impenetrable. Assuming no local spin of a fluid element, the water flow has to be irrotational, that is,

$$u_y = v_x. \tag{6}$$

In addition to the governing equations (1)-(6) for irrotational water waves [13], for a solitary wave we must have that, as $x \to \pm\infty$, the flow is at rest and the free surface approaches a height $h > 0$ above the flat bed. The parameters $c > 0$ and $h > 0$ cannot be arbitrarily chosen: given the wave speed $c > 0$, the inequality

$$c > \sqrt{gh} \tag{7}$$

must hold for nontrivial solutions (see [1]). Moreover, all solitary waves are *a priori* of positive elevation above their asymptotic limit h, symmetric about a single crest and with a strictly monotone wave profile on either side of this crest, as shown by Craig and Sternberg [8].

The existence of small amplitude solitary waves was proved by Friedrichs and Hyers [11] via a power series method, while Beale [4] used the implicit function theorem of Nash-Moser type to improve this result. The existence of large amplitude solitary waves, including the existence of a solitary wave of greatest height, is due to Toland and collaborators (see [1, 2, 3, 12]) and relies on global bifurcation theory. We now present some fundamental properties of solitary waves established in these papers, and we derive some conclusions that are relevant for our purposes. To facilitate our task, assuming that at time $t = 0$ the wave crest is located at $x = 0$, let

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \ 0 < y < \eta(x)\}$$

be the fluid domain at time $t = 0$, with its two components

$$\Omega_- = \{(x,y) \in \mathbb{R}^2 : \ x < 0, \ 0 < y < \eta(x)\}$$

and

$$\Omega_+ = \{(x,y) \in \mathbb{R}^2 : \ x > 0, \ 0 < y < \eta(x)\},$$

the boundaries (top and bottom) of which we denote by

$$S_- = \{(x,y) \in \mathbb{R}^2 : \ x < 0, \ y = \eta(x)\}, \quad B_- = \{(x,y) \in \mathbb{R}^2 : \ x < 0, \ y = 0\},$$

respectively

$$S_+ = \{(x,y) \in \mathbb{R}^2 : \ x > 0, \ y = \eta(x)\}, \quad B_+ = \{(x,y) \in \mathbb{R}^2 : \ x > 0, \ y = 0\}.$$

Defining the stream function $\psi(x,y)$ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v, \tag{8}$$

we see that ψ is harmonic in Ω in view of (6), whereas (4) and (5) ensure that ψ is constant on both boundaries of Ω, say $\psi = 0$ on $y = \eta(x)$ and $\psi = m$ on $y = 0$. Throughout Ω we have $u < c$, with the inequality extending to the boundary of Ω (the free surface is the graph of a real-analytic function and the fluid velocity components have harmonic extensions across it, except for the wave of greatest height, where the curve is real-analytic everywhere but at the crest, where it is just continuous with a corner containing an angle of $2\pi/3$), except for the wave of great height, in which case $u = 0$ at the crest with $u < c$ elsewhere (see [1, 3]). That is,

$$\psi_y = u - c < 0 \quad \text{in} \quad \Omega \cup S_- \cup S_+ \cup B, \tag{9}$$

where $B = \{(x,0) : \ x \in \mathbb{R}\}$ is the flat bed, so that $m > 0$. Using the implicit function theorem we deduce that for all $\alpha \in (0, m]$ the level curve $\{\psi = \alpha\}$ is a smooth curve $y = h_\alpha(x)$. Notice that $h_m \equiv 0$, while h_0 is the profile of the wave (with a corner at $x = 0$ in the case of the wave of greatest height). Following some ideas due to Craig and collaborators in conjunction with the insight provided by the results of Toland and collaborators, we now prove some useful facts.

Lemma 1. *At any given time t the horizontal velocity component u is positive, while the sign of the vertical velocity component v at a point in the fluid depends on the position of the point with respect to the crest: $v = 0$ below the crest and on the flat bed, $v > 0$ if the crest is behind the particle located above the bed, and $v < 0$ if the point is above the bed and behind the crest.*

Proof. Since with respect to a frame of reference moving with speed c the flow is steady and occupies a fixed region, it suffices to prove that $u > 0$ throughout $\overline{\Omega}$ and $v(x,y) > 0$ if $(x,y) \in \Omega_+ \cup S_+$, while $v(x,y) < 0$ if $(x,y) \in \Omega_- \cup S_-$.

Notice that $v = 0$ on the flat bed $B = \{(x,0) : \ x \in \mathbb{R}\}$ in view of (5). The symmetry properties established by Craig and Sternberg [8] ensure that u and η are symmetric with respect to the line $\{x = 0\}$, while v is anti-symmetric. In particular, $v(0,y) = 0$ for all $y \in [0, \eta(0)]$. Notice that the convergence $u, v \to 0$ as $|x| \to \infty$ is exponentially fast (see [1]). Furthermore, since (4) and (9) ensure that $v < 0$ on S_- as the profile $x \mapsto \eta(x)$ of the free surface is strictly increasing for $x < 0$, while $v = 0$ at $x = -\infty$ and on B_- as well as for $x = 0$, we deduce by the maximum principle for the harmonic function v in Ω_- that $v < 0$ in Ω_-. Similarly one shows that $v(x,y) > 0$ for $(x,y) \in \Omega_+ \cup S_+$.

To prove that $u > 0$ in $\overline{\Omega}$ we proceed as follows. Notice that (2) yields Bernoulli's law: throughout the fluid domain Ω the expression $\frac{(u-c)^2+v^2}{2} + P + gy$ is constant. Evaluating the expression on the free surface as $|x| \to \infty$, we deduce that this constant value equals $\frac{c^2}{2} + P_0 + gh$. Consequently

$$\frac{(u-c)^2 + v^2}{2} + P + gy = \frac{c^2}{2} + P_0 + gh \quad \text{throughout} \quad \Omega. \tag{10}$$

On the other hand, a direct calculation based on (2) and (8) yields that P is superharmonic in Ω:

$$\Delta P = -2\psi_{xy}^2 - 2\psi_{xx}^2 \leq 0.$$

Therefore, in view of (3), the minimum of P in $\overline{\Omega}$ is attained on the flat bed or on the free surface since (10) yields

$$P(x,y) \to P_0 + g(h-y) \geq P_0 \quad \text{for} \quad |x| \to \infty.$$

But $P_y = -g$ on B by (2) and (5) so that Hopf's maximum principle [10] ensures that the minimum of P is attained everywhere on the free surface (where $P = P_0$) and $P > P_0$ in Ω. Again by Hopf's maximum principle, we infer that $P_x(x,\eta(x)) > 0$ for $x < 0$, while $P_x(x,\eta(x)) < 0$ for $x > 0$ if we take into account (3) and the strict monotonicity of the graphs S_- and S_+. But $P_x = (c-u)[u_x + \eta_x u_y]$ on $y = \eta(x)$ in view of (2) and (4), so that (9) ensures $u_x(x,\eta(x)) + \eta_x(x)\, u_y(x,\eta(x)) > 0$ for $x < 0$, with the opposite sign for $x > 0$. That is,

$$\text{(11)} \qquad \frac{d}{dx}\, u(x,\eta(x)) > 0 \quad \text{for} \quad x<0, \quad \text{and} \quad \frac{d}{dx}\, u(x,\eta(x)) < 0 \quad \text{for} \quad x > 0.$$

In other words, along the free surface u increases strictly from $x = -\infty$ to the crest $x = 0$, and thereafter it is strictly decreasing. But the function u is harmonic in Ω and on the flat bed B we have $u_y = v_x = 0$ by (5) and (6), so that Hopf's maximum principle ensures that the minimum and maximum of u cannot occur on the bed. Since $u \to 0$ as $|x| \to \infty$, the monotonicity properties of u along the curve $y = \eta(x)$ encompassed in (11) ensure that $u > 0$ throughout $\overline{\Omega}$, with u reaching its maximum value (less or equal to the wave speed c) at the crest $(0,\eta(0))$. $\square$

Lemma 2. *To each fluid particle moving within the water there corresponds a unique time $t^* \in \mathbb{R}$ so that at $t = t^*$ the particle is exactly below the wave crest, while afterwards it is located behind the wave crest, the wave crest being behind the particle for $t < t^*$.*

Proof. The path (past and future) $(x(t), y(t))$ of a particle with location $(x(0), y(0))$ at time $t = 0$ is given by the solution of the differential system

$$\text{(12)} \qquad \begin{cases} x' = u(x-ct, y), \\ y' = v(x-ct, y). \end{cases}$$

Associate to (12) the Hamiltonian system

$$\text{(13)} \qquad \begin{cases} X' = u(X,Y) - c, \\ Y' = v(X,Y) \end{cases}$$

in the moving frame

$$\text{(14)} \qquad X = x - ct, \quad Y = y,$$

in which the wave is stationary. The Hamiltonian function for (13) is $\psi(X,Y)$ in view of (8). Notice that $X(t)$ describes precisely the position of the particle with respect to the wave crest at time t, assuming that initially (at time $t=0$) the wave crest is located at $x = 0$: $(X(t), Y(t)) \in \overline{\Omega}$. All solutions of (13) are defined globally in time (in the past and in the future), since the boundedness of the right-hand side prevents blow-up in finite time.

If we do not deal with a wave of greatest height, then for some $\varepsilon > 0$ we have that $u \leq c - \varepsilon$ throughout Ω. Therefore $X(t) \leq x(0) - \varepsilon\, t$ for $t > 0$, with $X(t) \geq x(0) + \varepsilon\, t$ for $t < 0$, and the statement of the lemma follows at once.

Let us now address the issue in the case of the wave of greatest height, in which case $u = c$ at the wave crest, with $u < c$ elsewhere in the fluid. Taking into account (9), we find that $X'(t) \leq 0$ with equality sign possible only if $X(t) = 0$ and $Y(t) = \eta(0)$. Notice that $(0, \eta(0))$ is the only critical point of the continuous autonomous system (13) with a right-hand side that is uniformly bounded in $\overline{\Omega}$ and smooth everywhere except at the critical point. Since $Y(t) = h_\alpha(X(t))$ for some $\alpha \in [0, m]$ as $\psi(X(t), Y(t)) = \alpha$ for all $t \in \mathbb{R}$, we deduce that the only two possible scenarios for the statement not to hold are that $(X(0), Y(0)) \in S_+ \cup S_-$ reaches $(0, \eta(0))$ in infinite time or $(X(t), Y(t)) = (0, \eta(0))$ for all $t \in [0, T]$ with some $T > 0$. The first possibility cannot occur, since for, say, $(X(0), Y(0)) \in S_+$ we have $Y(0) = h_0(X(0))$ so that $Y(t) = \eta(X(t))$ for all $t \geq 0$, and our claim follows as the inequality

$$\int_0^{X(0)} \frac{dx}{c - u(x, \eta(x))} < \infty \tag{15}$$

shows that the decreasing function $X(t)$ reaches 0 in finite (positive) time. The inequality (15) follows from the fact that $c - u(x, \eta(x)) = O(\sqrt{x})$ as $x \downarrow 0$. This last estimate is a direct consequence of the fact that $\eta'(x)$, existing for $x \neq 0$, is bounded away from zero as $x \downarrow 0$. Indeed, (3) and (10) yield

$$[c - u(x, \eta(x))]^2 + v^2(x, \eta(x)) = c^2 + 2g[h - \eta(x)] = O(x) \quad \text{as} \quad x \downarrow 0, \tag{16}$$

with the decay rate a consequence of the mean value theorem as $\eta(0) = h + \frac{c^2}{2g}$ for the wave of greatest height in view of (10). Furthermore, notice that the corner at the crest for the wave of greatest height contains an angle of $2\pi/3$. Therefore, if $\theta(x)$ is the angle between the wave profile $x \mapsto \eta(x)$ and the horizontal direction $(1, 0)$, we have $\lim_{x \uparrow 0} \theta(x) = \pi/6$ while $\lim_{x \downarrow 0} \theta(x) = \pi - \pi/6$. But

$$\tan(\theta(x)) = \eta_x(x) = \frac{v(x, \eta(x))}{u(x, \eta(x)) - c} \quad \text{for} \quad x \neq 0$$

if we take into account (4). In conjunction with (16), the above inequality yields

$$c - u(x, \eta(x)) = O(\sqrt{x}) \quad \text{and} \quad v(x, \eta(x)) = O(\sqrt{x}) \quad \text{for} \quad |x| \downarrow 0.$$

In particular, (15) holds true. This means that in the case of the wave of greatest height uniqueness fails for the solution of (13) with initial data $(0, \eta(0))$. The possibility that a particle might stay at the wave crest for a positive period of time is ruled out by the following reasoning. The physically reasonable solution of (13) with initial data $(0, \eta(0))$ is not the constant solution, since this would mean that particles collide at the crest; otherwise the particle located at the crest would move at constant speed $c > 0$ to the right, whereas, in view of (15), any particle located initially at some point $(x(0), y(0))$ with $x(0) > 0$ and $y(0) = \eta(x(0))$ would reach the crest in finite time. Therefore in the moving frame a solution starting on $y = \eta(x)$ reaches the point $(0, \eta(0))$ in finite time and does not pause there but moves on with a decreasing X-coordinate as time goes by. This concludes the proof in the case of the wave of greatest height. □

Remark. The wave of greatest height has a stagnation point at its crest since $u = c$ and $v = 0$ there. Lemma 2 shows the somehow counter-intuitive fact that this nevertheless does not mean that the particle at the crest moves along with the wave at speed c. Instead, the particle at the crest is left behind as the wave moves on.

We conclude this section by recalling that the streamlines $\psi(x,y) = \alpha$ with $(x,y) \in \overline{\Omega}$ and $\alpha \in [0,m]$ fixed are smooth curves $y = h_\alpha(x)$, except for h_0 in the case of the wave of greatest height. Due to the exponential decay of (u,v) as $|x| \to \infty$, we deduce that for each fixed $y_0 \in [0,\eta(0)]$ the streamline $y = h_\alpha(x)$ with $\alpha = \psi(0,y_0)$, passing through the point $(0,y_0)$, has an asymptote $y = l(y_0)$ as $|x| \to \infty$, with $l(\eta(0)) = h$ and $l(0) = 0$. Furthermore, the function $x \mapsto h_\alpha(x)$ is strictly decreasing on $(0,\infty)$ and strictly increasing on $(-\infty,0)$ for all $\alpha \in [0,m)$, while $h_m(x) = 0$ for all $x \in \mathbb{R}$.

3. Main result

We now describe the particle trajectories in a solitary wave; see Figure 2.

Theorem. *A particle on the flat bed moves in a straight line to the right at a positive speed. Any particle above the bed reaches at some instant t^* the location (x_0,y_0) below the wave crest $(x_0,\eta(0))$. The particle trajectory is confined for $t \neq t^*$ to the region strictly below the streamline $y = h_\alpha(x)$ in $\overline{\Omega}$, where $\alpha = \psi(0,y_0)$, and strictly above the asymptote $y = l(y_0)$ at $x = \pm\infty$ of this streamline. As time t runs on $(-\infty,t^*)$, the particle moves to the right and upwards, while for $t > t^*$ the particle moves to the right and downwards.*

Proof. For particles on the bed the statement is already proved in Section 2. For a particle located above the flat bed, without loss of generality, in view of Lemma 2, we may assume that $t^* = 0$ and $x_0 = 0$. Let $(x(t),y(t))$ be the corresponding particle trajectory with $y(0) = y_0 > 0$ and $x(0) = 0$. It suffices to prove the statement for $t > 0$, as the case $t < 0$ can be dealt with similarly. In view of Lemma 1, we deduce that $x(t)$ is strictly increasing for $t > 0$. On the other hand, (8) and (12) yield

$$\frac{d}{dt}\,\psi(x(t),y(t)) = -c\,v(x(t)-ct,y(t)) > 0, \qquad t > 0,$$

with the inequality justified by Lemma 1 since $x(t) - ct < 0$. Therefore

$$\psi(x(t),y(t)) > \psi(0,y_0) = \alpha \quad \text{for} \quad t > 0, \tag{17}$$

and (9) ensures that for $t > 0$ the particle trajectory lies strictly below the streamline $y = h_\alpha(x)$. Furthermore, Lemma 1 and (12) ensure that $y(t)$ is strictly decreasing for $t > 0$. We claim that $\lim_{t\to\infty} y(t) = l(y_0)$. Indeed, (17) and the monotonicity of the streamline $y = h_\alpha(x)$ ensure that $\lim_{t\to\infty} y(t) \le l(y_0)$. On the other hand, since ψ is the Hamiltonian function of the system (13), we have

$$\psi(x(t)-ct,y(t)) = \psi(0,y_0) = \alpha \quad \text{for} \quad t \ge 0,$$

which means that $h_\alpha(x(t)-ct) = y(t)$ for all $t \ge 0$. Therefore

$$l(y_0) = \lim_{|x|\to\infty} h_\alpha(x) \le \limsup_{t\to\infty} h_\alpha(x(t)-ct) = \lim_{t\to\infty} y(t).$$

With the opposite inequality already established, we infer that $\lim_{t\to\infty} y(t) = l(y_0)$, and the proof is complete. □

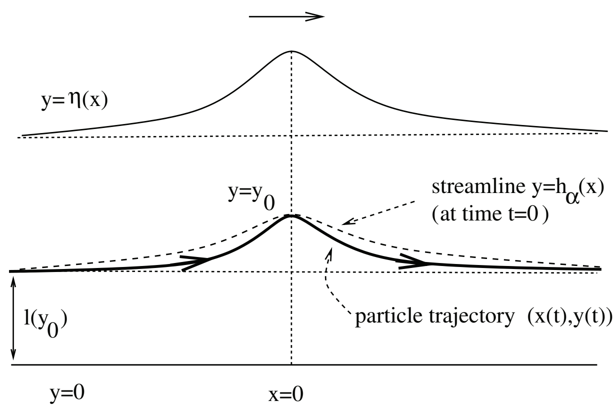

Figure 2. Just like the free surface, each streamline moves to the right with speed c. The streamlines at time $t = 0$ determine the location of particle trajectories even if the particles do not move on streamlines.

4. Comments

Recently Craig [7] proved that the solitary wave phenomenon can occur only for two-dimensional water waves: for the three-dimensional water wave problem there do not exist any localized steady waves of elevation propagating over a flat bed.

It is of interest to compare the particle trajectories in solitary waves with the particle paths in periodic waves, especially since it is known (see [2]) that periodic waves converge to solitary waves in the long-wave limit. Interestingly, while in periodic waves within a period each particle experiences a backward-forward motion with a slight forward drift (see [5]), we saw that in a solitary water wave there is no backward motion: all particles move in the direction of wave propagation at a positive speed, the direction being upwards/downwards if the particle precedes or does not precede the wave crest. This shows that in the long-wave limit the shapes of the periodic waves approach the profile of a solitary wave but the pattern of the particle motion within the fluid is not preserved in this limiting process.

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