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EULER'S DEFINITION OF THE DERIVATIVE

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Abstract. Euler's method of defining the derivative of a function is not a failed effort to describe a limit. Rather, it calls for rewriting the difference quotient in a way that remains meaningful when the denominator is zero.

I was once sitting next to a Lebanese friend at a concert of chamber music in which the ensemble included a lute. "That's an oud she has," he said, indicating the lutist, "but she doesn't know how to play it!" He was joking. He knew perfectly well that the lute is closely related to a Middle Eastern instrument called the oud, and he knew perfectly well that in Europe it is used to play a very different type of music. I am reminded of his joke when I read descriptions by present-day mathematicians of Euler's conception of the derivative. "He's talking about a limit," they say, "but he doesn't know how to do it!"

Euler's undisputed greatness entitles him to a fair hearing when modern writers[1] tell us that he had a weak grasp of the fundamental notions of the calculus and that it was only after Cauchy, Weierstrass, and others formulated the limit concept in terms of epsilon and delta that calculus became well founded. He set "geometric rigor" as his goal,[2] and we should not readily believe that he faked his way through calculus with a poor grasp of its basic concepts and a casual attitude toward mathematical reasoning.

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[1]For example, when Morris Kline in [5, p. 429] describes Euler's reasoning about the definition of the derivative, he puts the word reasoning in quotes, implying that he regards it as no reasoning at all. He also says, p. 948, that "the underlying concepts of limit, continuity, and so on" are not in the work of Lagrange. These "underlying concepts" are also absent from Euler, so if they are essential—truly underlying—Euler's work does not get to the heart of the matter. Dirk Struik [6, p. 176] says, "We cannot . . . be enthusiastic about Euler's basing the calculus on the introduction of zeros of different orders" which seems to me to misrepresent Euler's conception of the calculus. Dunham [2, p. 54] describes what he says is Euler's derivation of the rule $\frac{d}{dx}\sin x = \cos x$ and says "Nothing to it!" meaning, I believe, that the derivation is little more than a heuristic argument. I agree with the first part of Boyer's statement [1, p. 492], "Inasmuch as Euler restricted himself to well-behaved functions, he had not become involved in the subtle difficulties that later were to make his naïve position untenable," but not with the second. The fact that real numbers cannot be added by putting them over a common denominator does not make the ordinary addition of fractions "untenable."

[2]In the seventh paragraph of his introduction to *Institutiones Calculi Differentialis* he writes: "If someone undertakes to calculate the magnitude of the whole terrestrial globe, it is the custom easily to grant him an error not only of a single grain of dust, but of even many thousands of these. However, geometric rigor shrinks from even so small an error, and this objection would be simply too great were any force granted to it."

Although Euler is generally credited today—and correctly so—with basing calculus on the function concept and with stressing that a function need not be given by a formula, his definition of the derivative is misunderstood primarily because his notion of "function" is misunderstood.

In the introduction to his book *Institutiones Calculi Differentialis* (Foundations of Differential Calculus) [3], he describes calculus as dealing with *quantities.* Some quantities vary and some are constant. Differential calculus studies the relations among the variations of the variable quantities. He illustrates the types of quantities and relations he has in mind with quantities related to the firing of a cannon—the amount of gunpowder, the elevation of the cannon, the distance traveled by the shot, the length of time the shot is in the air, and so forth. The relations among these quantities are not thought of as being given by formulas, but on the other hand they are surely not thought of as being the sort of general set-theoretic, anything-goes subsets of product spaces that modern mathematicians mean when they use the word "function". The underlying, and, it must be admitted, unexpressed assumption is that the relations among the varying quantities are such that they can be analysed, meaning that they are embedded in a coherent system of quantities that can be studied mathematically.

To avoid this confusion of what Euler meant by a function with the modern-day notion, I will just say that Euler studied *systems of related variable quantities.* For example, the rational functions on a given algebraic curve are related variable quantities or, in the firing of a cannon, the quantities Euler cites.

Let x and y be related variable quantities, and let $x = a$ and $y = b$ be values they assume, near which y is a function of x—each value of x near a determines a value of y near b. Then $x - a$ and $y - b$ are also variable quantities as is their ratio. Euler insists that although $x - a$ and $y - b$ are both zero when x is a, their *ratio* has, or at any rate can have, a value when $x = a$.

I read this definition for the first time a year or two ago. It happened when I was astonished by the quotations from Euler provided by a modern writer who was describing what he clearly regarded as Euler's unsuccessful attempt to define the derivative. I felt I had to see whether the original whole could be as incomprehensible as the quoted parts. When I understood enough of the context to realize what Euler was saying, I experienced a shock of recognition. It was practically the same as the definition of the derivative that I finally chose after decades of teaching calculus: "Rewrite $\frac{\Delta y}{\Delta x}$ in a way that still makes sense when $\Delta x = 0$."

Everyone's favorite example—Euler's too—is $y = x^2$. In his description of this example Euler writes ω instead of Δx, and he observes that when x becomes $x+\omega$, y becomes $(x + \omega)^2 = x^2 + 2x\omega + \omega^2$; the change in x is ω and the change in y is $2x\omega+\omega^2$, so the *ratio* of the changes is $2x+\omega$. When x is a, ω is zero and the ratio is $2a$. Both $x-a$ and $y-b$ have the value 0 at the point in question, but their ratio has the value $2a$.

Of course Euler understood limits. Euler was Euler. But he rejected limits as the way to define derivatives. The derivative was not, for him, about the way that Δy and Δx vanished. He insisted—and at some length—that no quantity is infinitely small without being zero. But this in no way prohibits a ratio of two variable quantities from having a value at a point at which both quantities are zero, because a ratio of variable quantities is itself a variable quantity.

He makes no claim that this ratio can always be evaluated. "As differential calculus is extended to all kinds of functions, no matter how they are produced, it is not immediately known what method is to be used to compare the vanishing increments of absolutely all kinds of functions."[3] His claim is simply that there is a rigorous meaning to what one is attempting to do.

This formulation explains the organization of Euler's six volume work on the calculus—very peculiar from a modern point of view—in which the definition of the derivative comes only in the third volume and the definition of the integral only in the fourth. What could fill those first two volumes? They are called *Introductio in Analysin Infinitorum* (Introduction to the Analysis of Infinites) [4], and they deal largely with the *transformation of functions,* that is, expressing relations between variable quantities in different ways. The first volume is largely about *series expansions.* The second deals with algebraic curves. In connection with tangents, normals, and so forth, he says in his Preface, "Although all of these nowadays are ordinarily accomplished by means of differential calculus, nevertheless, I have here presented them using only ordinary algebra, in order that the transition from finite analysis to analysis of the infinite might be rendered easier." Thus, when he arrives at derivatives in the third volume, *Institutiones Calculi Differentialis,* he has prepared the reader very well to express $\frac{\Delta y}{\Delta x}$ in different ways.

And that was his purpose. The first two sentences of his preface to the first volume of *Introductio* make that clear:

> Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art. From this it follows not only that they remain on the fringes, but in addition they entertain strange ideas about the concept of the infinite, which they must try to use.[4]

Anyone who has tried to explain the derivative of x^n to a class in which the students profess ignorance of the formula $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + 1)$ will know what he means.

But I wish Euler had specified the "strange ideas about the concept of the infinite" he was referring to. In that way he would have revealed something about his own ideas of the infinite. At first glance, his reputation for being careless about the use of infinity in mathematical arguments sometimes seems deserved, but this judgement may be superficial. Perhaps he simply felt that if ideas were presented in their proper order and his readers became well acquainted with elementary uses of infinity, they would develop the correct instinct for how to proceed in such arguments without his having to go into rigorous justifications that would be more baffling than enlightening.

Consider, for example, his derivation of the development of $\sin x$ in powers of x. He uses the formula

$$(\cos y + i \sin y)(\cos z + i \sin z) = \cos(y+z) + i \sin(y+z),$$

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[3]Quoted from the Preface of the English translation [3].

[4]Quoted from [4]. The original Latin is: *Saepenumero animadverti, maximam difficultatum partem, quas Matheseos cultores in addiscenda Analysi infinitorum offendere solent, inde oriri, quod, Algebra communi vix apprehensa, animum ad illiam sublimiorem artem appellant; quo fit, ut non solum quasi in limine subsistant, sed etiam perversas ideas illius infiniti, cuius notio in subsidium vocatur, sibi forment.*

which can be regarded as a convenient formulation of the addition formula for sines and cosines, in the case of a large number N of equal factors $\cos z + i \sin z$ to find that $\sin(Nz)$, which is the imaginary part of $(\cos z + i \sin z)^N$, is

$$\sin(Nz) = \frac{N}{1} \cdot \cos^{N-1} z \cdot \sin z - \frac{N(N-1)(N-2)}{1 \cdot 2 \cdot 3} \cdot \cos^{N-3} z \cdot \sin^3 z + \frac{N(N-1)(N-2)(N-3)(N-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cos^{N-5} z \cdot \sin z^5 - \cdots \tag{0.1}$$

(a terminating series) by virtue of the binomial theorem. The expansion of $\sin x$ in powers of x is simply, Euler tells us, the case of this formula in which z is infinitely small, N is infinitely large, and Nz is finite, say $Nz = x$. Indeed, if we agree that $\cos z = 1$ when z is infinitely small while $\sin z = z$ (by the definition of radian measure as we would say today), one finds immediately that

$$\begin{aligned} \sin x &= N \cdot \frac{x}{N} - \frac{N(N-1)(N-2)}{1 \cdot 2 \cdot 3} (\frac{x}{N})^3 \\ &\quad + \frac{N(N-1)(N-2)(N-3)(N-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\frac{x}{N})^5 - \cdots \\ &= x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots \end{aligned} \tag{0.2}$$

when N is infinite.

Do not scoff. We would have no quarrel with this argument if he said "in formula (0.1) set $z = \frac{x}{N}$ to find

$$\sin x = \cos^{N-1} \frac{x}{N} \cdot N \sin \frac{x}{N} - \frac{(N-1)(N-2)}{1 \cdot 2 \cdot 3 \cdot N^2} \cos^{N-3} \frac{x}{N} \cdot N^3 \sin^3 \frac{x}{N} + \frac{(N-1)(N-2)(N-3)(N-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot N^4} \cos^{N-5} \frac{x}{N} \cdot N^5 \sin^5 \frac{x}{N} - \cdots \tag{0.3}$$

and let $N \to \infty$ using $\lim_{N\to\infty}(N \cdot \sin \frac{x}{N}) = x$," and, after all, that is about what he does say. We would only want to know how he justifies passing to the limit termwise. Euler does not enter into any discussion or justification of the interchange of limits, but he does not allow his readers to ignore the issue. On the contrary, he sets $x = \frac{m}{n} \cdot \pi$ and gives the *numerical values to 28 decimal places* of the coefficients of the *first 15 terms* of the series in $\frac{m}{n}$ that results! He says he does this to show the reader how the formula is to be used, from which I infer that he intended his readers to use it. If one can be satisfied with a mere 12 decimal places of accuracy, one can then find $\sin 1^\circ$ using only 3 terms of the power series whose coefficients Euler provides, with $\frac{m}{n} = \frac{1}{180} = .005555\ldots$, to find the (correct) answer .017452406437 (rounded). In the process, the reader learns that the formula *can be* used—the convergence is fairly rapid even at the upper end $\frac{m}{n} = \frac{1}{2}$ of the range where it is needed. (In fact, he *also* gives the coefficients of the corresponding series for $\cos x$, so the range $0 \leq \frac{m}{n} \leq \frac{1}{4}$ suffices.)

I have sometimes thought that the existence of calculators and computers gives modern students an advantage because the calculations that calculus explains have become easy to do. At other times I think modern students regard calculation as something that is done by machines and that this abdication of the tasks of calculation cuts them off from the kind of thinking needed to understand calculus. Seeing what Euler did and expected his readers to consider doing in the way of calculation makes me think that the second view is the correct one. You have to walk before you can run, but what if you never get out of your car?

In any event, it seems to me that rather than avoiding serious questions of passage to limits and convergence, Euler is giving his readers a stock of experience with specific examples that will make the operations of calculus have real meaning for them. Certainly if a reader has a good enough understanding of the calculation of $\sin 1^\circ$ to see the issues raised by the truncation of the series or the use of a value of N that is not large enough, he or she will have little difficulty in providing the needed error estimates.

Since the definition of the derivative is still two volumes in the future, Euler does not use his series for sine and cosine to find derivatives, but the series expansion of $\sin x$ of course yields $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots$ and the analogous expansion of $\cos x$ yields $\frac{\cos x - 1}{x} = -\frac{x}{2} + \frac{x^3}{24} - \cdots$, so these variable quantities have the values 1 and 0, respectively, when $x = 0$. Thus, the ratio of the differences $\sin(x+\omega) - \sin x$ and $(x+\omega) - x$, which is $\frac{\sin x \cos \omega + \cos x \sin \omega - \sin x}{\omega} = \sin x \cdot \frac{\cos \omega - 1}{\omega} + \cos x \cdot \frac{\sin \omega}{\omega}$, is $\cos x$ when $\omega = 0$. Similarly,[5] $\cos(x+\omega) - \cos x$ divided by $(x+\omega) - x$ is $-\sin x$ when $\omega = 0$.

Only after two volumes of such work involving many examples and much algebraic preparation does Euler come to the definition of the derivative as the "ratio of two vanishing increments." He is not making an unsuccessful attempt to define the derivative as a limit in the modern way. Instead, he is applying the methods he has developed in the first two volumes to find the value of $\frac{\Delta y}{\Delta x}$ when $\Delta x = 0$. Whether these methods can be regarded as rigorous is not a cut-and-dried question. If one requires a definition that is valid for the most general conceivable (differentiable) function, Euler's definition certainly will not do. For algebraic and exponential and trigonometric functions and other functions derived from combinations of them, it does very well and is not lacking in rigor. In addition, it is practical, it has simple and clear meaning, and it can be extended to many other transcendental functions. Is there any reason Euler should have regarded it as unsatisfactory and waited for Cauchy or Weierstrass to explain limits to him?

The next time you read about Euler's inability to deal rigorously with the notions of the calculus, I hope you will remember that it is not a twentieth-century set-theoretic lute that he is playing, but a fine eighteenth-century analytic oud. And he is making the sort of constructive mathematical music that some of us in the twenty-first century like to hear.

About the author

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[5]It is true that when Euler does derive the formulas for the derivatives of $\sin x$ and $\cos x$ in Section 201 of *Institutiones Calculi Differentialis,* he does not put it quite this simply, probably because he regards it as so clear that he wants to use it to illustrate computation with differentials. (See Section 195, where he also mentions it in passing.) Such computation with differentials—differential calculus—is his subject in this volume, but it will not find a receptive audience today, when students are taught to shrink from differentials as from an infectious disease.

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