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The monodromy group, by H. Zoladek, Birkhäuser, Basel-Boston-Berlin, 2006, xi+ 580 pp., US$129.00, ISBN 978-3-7643-7535-5

This book covers various topics of modern analysis and geometry related to the concept of monodromy: singularity theory, local and global theory of analytic differential equations, Abelian integrals, differential and topological Galois theory, multiple hypergeometric integrals. We mainly concentrate in this review on differential equations and related topics. The progress of the last three decades has drastically changed the panorama of the field. In this review, together with major results, some open problems are presented.

1. Linear systems with regular singular points

For a long time, the Riemann-Hilbert problem was a central one in the theory of linear differential equations with complex time. This problem requires the construction of a linear differential equation with a preassigned monodromy data. The monodromy data is defined as follows. A circuit around a singular point of the equation produces a linear transformation of the space of its solutions, called *a monodromy map.* The set of singular points $(a_1, ..., a_m)$ and corresponding monodromy maps $(M_1, ..., M_m)$ are called *the monodromy data.*

Riemann, who discovered the very concept of monodromy, stated the problem at the end of his short life. Poincaré and Hilbert tried to solve it, and Hilbert included it in his famous list under the number 21. In 1908 J. Plemelj, a disciple of Hilbert, solved the problem. He presented his solution in full detail in a book that he published in 1964, at the age of 89. For about 70 years the mathematical community believed that the problem was completely solved.

Only in the 1970s did it become clear that Plemelj realized arbitrary monodromy data for *regular,* not for *Fuchsian*, systems. A linear system

$$\dot{z} = A(t)z, \;\; z \in \mathbb{C}^n \tag{1}$$

is *regular* provided that A is rational and the solutions have but a power growth at the singular points, the poles of A. The Fuchsian systems are those for which A has simple poles only:

$$\dot{z} = \sum_1^m \frac{A_j}{t - a_j} z; \tag{2}$$

$A_1, ..., A_m$ are called *the residue matrices* of the system (2). In 1989 A. Bolibrukh constructed a ground-breaking example of the monodromy data that cannot be realized by Fuchsian systems. A problem arose: *What data may be realized?* The answer depends on n. For $n = 2$ it is: any data. For $n = 3$, the criterion of realizability was obtained by Bolibrukh and Anosov, [AB]; for $n = 4$, by Bolibrukh and Gladyshev, [G]. For $n = 5$, the answer is unknown.

A simplified version of the problem is suggested by Arnold: *What is a minimal codimension of the nonlinearizable monodromy data?*

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Another form of the Riemann-Hilbert problem is related to Lie group theory. Suppose that the monodromy matrices M_j belong to a Lie group G. *Is it possible to realize this monodromy data by a Fuchsian system whose residue matrices A_j belong to the Lie algebra $\mathfrak{g}$ of G?* Note that if $A_j \in \mathfrak{g}$, then $M_j \in G$.

2. Linear systems with irregular singular points

The local theory of irregular singular points is more complicated than in the regular case. The reason is that in the regular case the normalizing Taylor series converge, whilst in the irregular case they diverge as a rule.

In the regular case a classical normalizing transformation

$$w = H(t)z \tag{3}$$

with a nondegenerate holomorphic matrix function H is used to bring system (1) to a convergent normal form.

In the irregular case, system (1) near a singular point zero has the form

$$\dot{z} = \frac{A(t)}{t^{r+1}}z, \ z \in \mathbb{C}^n, \ r > 0. \tag{4}$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $A(0)$. Equation (4) is *nonresonant* provided that the λ_j's are pairwise distinct. In the nonresonant case equation (4) may also be transformed by (3) to a convergent normal form with the variables separated, but this time H in (3) is not a holomorphic matrix function, but rather a *holomorphic cochain.* By definition, this cochain is a $2r$-tuple of holomorphic matrix functions: $H = (H_1, \dots, H_{2r})$. Each function H_j is defined in an open sector S_j with vertex 0. Sectors $S_1, \dots, S_{2r}$, cyclically ordered, cover a punctured neighborhood of zero. Components H_j and H_{j+1} differ by an exponentially small increment in $S_j \cap S_{j+1}$; more precisely, $H_j(t) \to E$ as $t \to 0$ in S_j, and $\Phi_j(t) - E := H_j \circ H_{j+1}^{-1}(t) - E = O(\exp(-\frac{C}{|t|^r}))$ in $S_j \cap S_{j+1}$. The $2r$-tuple $(\Phi_1, ..., \Phi_{2r})$ is called *the coboundary* of the cochain H. The statement that (3) conjugates (4) with its normal form

$$\dot{w} = \frac{B(t)}{t^{r+1}}w, \ B(t) = \text{ diag } (b_1(t), \dots, b_n(t)) \tag{5}$$

means that H_j conjugates (4) and (5) in any sector S_j.

A functional cochain is not merely a tuple of holomorphic (matrix) functions, but rather an entity. For example, if one component H_j decreases faster than any power of t as $t \to 0$ in S_j, then all the components H_k are identically zero. This is a *Phragmen-Lindelöf theorem for cochains* [IKh].

Due to the linearity of the problem, the coboundary of the normalizing cochain may be expressed through constant linear operators called *Stokes operators.* These operators are invariants of the analytic classification of irregular singular points and cannot be determined by any finite jet of the vector field at a singular point. The occurrence of such invariants is called *the Stokes phenomenon.*

The Riemann–Hilbert problem has been considered for linear systems with irregular singularities since the time of Birkhoff. Recent progress is described in [BMM].

3. Nonlinear differential equations in the plane

The planar theory of analytic differential equations, both real and complex, has been developed much further than its multidimensional analog. The main problem in this field is Hilbert's 16th: *What may be said on the number and position of limit cycles of a real planar polynomial vector field of degree n?* This problem is not yet solved, but it motivates the development of many branches of the theory, including local ones.

One of the most famous local results is the Desingularization Theorem: *Isolated singular points of planar analytic vector fields,* however complex they are, *may be split into a finite number of elementary singular points by a finite number of blow-ups.* A blow-up is a map that may be locally defined as $(x, y) \mapsto (x, \frac{y}{x}.)$ A singular point is elementary provided that at least one of its eigenvalues is nonzero. The desingularization theorem, without a proof, was claimed by Bendixson in 1901. The centennial history of this theorem was completed in 2006, when the transparent proof due to Van der Essen was published in the book by Zoladek.

On the other hand, elementary singular points are in a sense simple. Their orbital analytic classification has been mainly completed. In the *nonresonant case* (the ratio λ of the eigenvalues is not zero or negative rational or natural or inverse to natural), a germ of a planar vector field is analytically equivalent to a linear one under some special condition on λ. The statement of this condition, and the proof of its sufficiency, is due to Bruno. Proof of the necessity, due to Yoccoz, was one of the results that gained him the Fields Medal in 1994.

The analytic classification of resonant singular points, saddle-nodes ($\lambda = 0$) and resonant saddles (λ rational negative), was obtained in the early 1980s by Martinet and Ramis. It relies heavily upon the classification of germs of conformal maps tangent to identity achieved by Écalle, Malgrange and Voronin in 1981. All these classifications have functional moduli. These moduli are *coboundaries of normalizing cochains.* These cochains occur in the nonlinear theory in the same way as in the theory of irregular singular points. The occurrence of these functional moduli is called *the Nonlinear Stokes phenomena*; see [I2].

One of the major problems of the local theory going back to R. Thom is *to give a complete analytic classification of germs of planar analytic foliations near a singular point.* Despite reasonable progress [L-N], [MS], the problem has not yet been solved. A similar problem is to give a *topological* classification of the same germs.

Nonlinear Stokes Phenomena and functional cochains were applied by Écalle [E] and Ilyashenko [I1] to prove that *polynomial vector fields in the real plane have but a finite number of limit cycles.* This is a partial answer to Hilbert's question.

4. Parameter depending Abelian integrals

Another problem closely related to Hilbert's 16th is a problem on the number of zeros of integrals of the form

$$I(t) = \int_{\gamma(t)} \omega. \tag{6}$$

Here ω is a 1-form with polynomial coefficients of degree m, $\gamma(t)$ is a real oval, that is, a compact component of a level curve $H = t$, of a real polynomial H in two variables of degree n. Real ovals of such polynomials form continuous families of closed curves bounded by critical level curves of H. Zeros of integral (6) correspond to limit cycles generated from the ovals of H by a perturbation

$$H + \varepsilon\omega = 0. \tag{7}$$

Note that ovals of H satisfy the equation $dH = 0$.

This gives rise to the following infinitesimal Hilbert problem: *give an upper estimate of the number of real zeros of integral* (7). Varchenko and Khovanski proved the existence of a uniform estimate for zeros of integral (7):

$$\forall m, n\ \exists V(m, n) \text{ such that } \#\{I = 0\} \leq V(m, n).$$

This is one of the first applications of the *fewnomial theory* originated by Khovanski [Kh].

Abelian integrals (7) may be extended to the complex domain as nonunivalent functions. Their ramification is described by the Picard-Lefschetz theorem. These integrals form a particular case of the so-called Gauss-Manin connection.

The main problem in the field is *to give an upper estimate of the number* $V(m, n)$. Numerous particular results have been obtained by L. Gavrilov, D. Novikov–Yakovenko, G. Petrov–Khovanski, Glutsyuk and others.

5. Global theory of planar polynomial foliations

This theory mainly describes the topological properties of generic polynomial foliations of the complex projective plane. These properties are drastically different from parallel properties of foliations of the real plane. Generic polynomial vector fields on the real plane have but a finite number of limit cycles; ω-limit sets of their orbits are either steady-state points or cycles; generic vector fields are structurally stable. Generic polynomial vector fields in the complex plane have a countable number of complex limit cycles, all their leaves are dense and the foliations are topologically rigid. Roughly speaking, a foliation $\mathcal{F}$ is *topologically rigid* provided that any foliation topologically equivalent to $\mathcal{F}$ is analytically equivalent to it.

The main problem in the domain is *to generalize these results to foliations in higher dimensions.* Some progress in this study is achieved in [G-M], [LR].

The analytic theory of differential equations is presented in detail in the forthcoming book [IYa]. This book and Zoladek's have a large intersection, but they are written in different styles and their symmetric difference is large as well.

6. Differential and topological Galois theory

Differential Galois theory, also called Picard-Vessiot theory, mainly studies extensions of the field K of rational functions by the components of solutions of linear and nonlinear systems of differential equations. It reduces the problem of solvability of these equations in quadratures to the study of the group of automorphisms of the extension that are identical on K. This group is called *the differential Galois group* of the extension or, by abuse of language, the Galois group of the differential equation itself.

In the 1980s Ramis studied the Galois group of a linear system (4) near an irregular singular point. He proved that Stokes operators of the equation belong to its Galois group. He also gave a complete description of the Galois group of the equation above in terms of its formal normal form, monodromy and Stokes operators. Zoladek's proof of this theorem makes use of the Phragmen-Lindelöf theorem for functional cochains.

In the last decade methods of the Galois theory of linear equations have been applied to the study of nonintegrability of nonlinear differential equations.

At the beginning of the 1960s, Arnold originated a topological version of Galois theory. He connected nonsolvability of algebraic equations in quadratures with the unsolvability of the monodromy group of the corresponding algebraic functions. Khovanski, a graduate student of Arnold at that time, created a topological version of the Picard-Vessiot theory for linear differential equations with complex time. At the same time, he extended this approach to the functions that are beyond the Picard-Vessiot theory, for instance, to those that have a countable dense set of ramification points on different leaves of the corresponding Riemann surface. Recently Khovanski extended this theory to analytic functions of several variables.

7. The book of Zoladek

In the book of Zoladek most of the results above are presented. Due to the huge amount of material, the exposition is very concentrated. Yet it is quite clear, with all the ideas described before the technical details. Some proofs are replaced by sketches. Many proofs are drastically improved by the author in comparison with the original sources. Numerous original results of the author are also presented.

The book also contains the core of singularity theory. This includes the classification of critical points of functions, in particular, results of Tougeron and Arnold, the monodromy theorem for the Milnor fibration, and asymptotics of oscillating integrals. These results were basically obtained in the 1960s and 1970s. The main part of these results is presented in [AGV1], [AGV2]. The presentation of Zoladek is more concentrated and provides a rapid introduction to some highlights of the theory.

The book contains all the preliminary material from algebra and topology necessary for the understanding of the other parts.

The last chapter deals with hypergeometric functions. It starts with the classical results on the Gauss hypergeometric equation and presents the Picard-Deligne-Mostow theory. The chapter concludes with the introduction to the Gelfand-Kapranov-Zelevinski-Varchenko theory of multivariable hypergeometric functions.

The book is an encyclopedia of various topics in geometry and analysis related to the concept of monodromy. It contains all the preparatory material and may be used for various graduate courses. On the other hand, it contains a lot of material for future research, especially in analytic differential equations and related topics. The book is a most valuable source in analytic differential equations and an excellent treatment of the singularity theory.

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