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Combinatorics of Coxeter groups, by Anders Björner and Francesco Brenti, Graduate Texts in Math., vol. 231, Springer, New York, 2005, xiii + 363 pp., US$74.95, ISBN 3-540-44238-3

1. Some history

Serious work on groups generated by reflections began in the nineteenth century. In 1852 Möbius determined the finite subgroups of $O(3)$ generated by isometric reflections on the 2-sphere (or equivalently, by orthogonal linear reflections on $\mathbf{R}^3$). He showed that the fundamental domain for such a group on the 2-sphere was a spherical triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, with p, q, r integers ≥ 2. Since the sum of the angles in a spherical triangle is greater than π, we must have $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$. For $p \geq q \geq r$ the only possibilities for (p,q,r) are $(p,2,2)$ for any $p \geq 2$ and $(p,3,2)$ with $p = 3$, 4 or 5. The last three cases are the symmetry groups of the Platonic solids (the tetrahedron, cube or dodecahedron, respectively). Subsequent work of Riemann and Schwarz on hypergeometric functions showed the existence of groups generated by reflections across the edges of triangles with angles integral submultiples of π in either the Euclidean or hyperbolic plane. By the end of the nineteenth century, in connection with their work on automorphic forms, Klein and Poincaré had studied other groups generated by isometric reflections across the edges of polygons (with 3 or more edges) in the hyperbolic plane.[1]

In the second half of the nineteenth century work also began on finite reflection groups on S^n for $n > 2$ (or equivalently, finite linear reflection groups on $\mathbf{R}^{n+1}$), generalizing Möbius' results for $n = 2$. The work developed along two lines. First, around 1850, Schläfli classified regular convex polytopes in $\mathbf{R}^{n+1}$ for $n > 2$. He showed that the symmetry group of such a polytope was a finite group generated by reflections and, as in Möbius' case, the projection of a fundamental domain to S^n was a spherical simplex with dihedral angles integral submultiples of π. Second, around 1890, Killing and E. Cartan classified complex semisimple Lie algebras in terms of their root systems. In 1925, Weyl showed that the group of symmetries of such a root system was a finite group generated by reflections.[2] This intimate connection with the classification of semisimple Lie groups cemented reflection groups into a central place in mathematics. The two lines of research were united by Coxeter [4] in the 1930's. Coxeter classified discrete groups generated by reflections on the n-dimensional sphere or Euclidean space.

The central example of a finite reflection group is the symmetric group S_{n+1} acting on $\mathbf{R}^{n+1}$ by permutation of coordinates. Transpositions act as orthogonal reflections across hyperplanes of $\mathbf{R}^{n+1}$. The diagonal line L in $\mathbf{R}^{n+1}$ is fixed by S_{n+1}. So, S_{n+1} acts on the orthogonal complement $L^{\perp}$, and $L^{\perp}$ can be identified with $\mathbf{R}^n$. The associated root system is type $\mathbf{A}_n$, and the associated complex Lie group is $SL(n+1,\mathbf{C})$. If we project the standard basis of $\mathbf{R}^{n+1}$ to $\mathbf{R}^n$ and take its convex hull, we get a regular n-simplex. This exhibits S_{n+1} as the group

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[1] In this paragraph and the next I have relied on the Historical Note of [2, pp. 249–257].

[2] It turns out that these two cases (symmetries of regular polytopes and symmetries of root systems) cover all finite reflection groups.

of symmetries of the regular n-simplex. A fundamental domain on $\mathbf{R}^{n+1}$ is the convex region $\{x \in \mathbf{R}^{n+1} \mid x_1 \geq \cdots \geq x_{n+1}\}$. Projecting this region to $\mathbf{R}^n$, we get a simplicial cone, and then intersecting with S^{n-1}, we get a spherical $(n-1)$-simplex.

The other n-dimensional space of constant curvature, besides the sphere and Euclidean space, is hyperbolic n-space $\mathbf{H}^n$. It also admits isometric reflections, and there are discrete groups generated by reflections on $\mathbf{H}^n$ with compact fundamental domains (at least for fairly small values of n; cf. [8]).

These geometric examples have several common features. Suppose W denotes the group and T the set of reflections in W. The fixed point set of a reflection t is a "hyperplane" H_t. (In the spherical case, a "hyperplane" means a great subsphere of codimension one, and, similarly, in the hyperbolic case, it means a totally geodesic subspace of codimension one.) A connected component of the complement of the union of the H_t, over all $t \in T$, is the interior of a convex polytope K. (Such a polytope is a *chamber.*) In the spherical case, K must be a simplex, and in the Euclidean case, a product of simplices. In the case of $\mathbf{H}^n$, there are many more possibilities for K (at least when $n \leq 4$). Let S denote the set of reflections across the codimension one faces of K. Then S generates W and every element of T is conjugate to an element of S. Moreover, W acts simply transitively on the set of chambers, and K is a strict fundamental domain for the W-action in the sense that K intersects each W-orbit in exactly one point. It follows that K can be identified with the space of orbits. If s and t are reflections across adjacent faces of K, these faces make a dihedral angle of π/m for some integer $m \geq 2$ and st is rotation by $2\pi/m$ around the codimension two subspace $H_s \cap H_t$. (This can be seen by thinking about the dihedral subgroup generated by s and t.) In particular, the order of st is m. The similarity between the example of the symmetric group and other finite reflection groups is one reason for the interest of combinatorialists in the subject of reflection groups.

Around 1960 Jacques Tits introduced the notion of an abstract reflection group, which he called a "Coxeter group". Tits considered pairs (W, S) with W a group and S a set of involutions which generate W so that the group has a presentation of the following form: the set of generators is S and the set of relations is $\{(st)^{m(s,t)}\}$, where $m(s,t)$ denotes the order of st and the relations range over all unordered pairs $s, t \in S$ with $m(s,t) \neq \infty$. The pair (W, S) is a *Coxeter system* and W is a *Coxeter group.* It is remarkable that this simple definition encapsulates all the essential properties of a reflection group. This was demonstrated in 1968 with the appearance of the wonderful volume of Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4, 5, et 6* (an English translation of which has appeared as [2]). In the first part of [2] the basic properties of Coxeter groups are developed in a purely group theoretic and combinatorial manner. This is also done in Chapter 1 of the book under review. The most convincing justification for thinking of Coxeter groups as abstract reflection groups is Tits' proof that any Coxeter system (W, S) can be realized as a group generated by linear (not necessarily orthogonal) reflections[3] across the faces of a simplicial cone in $\mathbf{R}^S$. Moreover, W acts properly on the interior $\mathcal{I}$ of a convex cone in $\mathbf{R}^S$, and the W-action on $\mathcal{I}$ has all the properties of

[3] This representation is called the "contragradient of the geometric representation" in [2] as well as in the book under review.

the geometric actions discussed previously. These facts are proved in [2] as well as in Chapter 4 of the book under review.[4]

2. What is in this book

The authors write in the Foreword to [1]:

> By "combinatorics of Coxeter groups" we have in mind the mathematics that has to do with reduced expressions, partial order of group elements, enumeration, associated graphs and combinatorial cell complexes, and connections with combinatorial representation theory.

2.1. **Reduced expressions.** One begins with the *word length* function $l : W \to \mathbb{N}$, defined by putting $l(w)$ equal to the smallest integer n such that $w = s_1 \cdots s_n$ with $s_i \in S$. The word $s_1 \cdots s_n$ is a *reduced expression* for w. The separation property of a reflection can be encoded in various ways as a combinatorial condition on words in S representing elements of W (e.g. the "Deletion Property" or the "Exchange Property"). The best result along this line is Tits' solution to the Word Problem (cf. [1, §3.3] or [3, pp. 49–51]). This says, first of all, that any reduced expression for a given $w \in W$ can be transformed into any other one by a sequence of moves which only involve replacing an alternating subword $st\cdots$ in s, t of length $m(s,t)$ by the alternating word $ts\cdots$ of the same length (such moves are called "braid moves" in [1]). Secondly, any expression for w can be shortened to a reduced expression by a sequence of braid moves and cancellations (or "nil moves") involving replacing a subword of the form ss by the empty word.[5]

Weak order and Bruhat order. There are two different partial orders on W. The first, called the *weak right order*, is defined by $v <_R w$ if there is a reduced expression for w which begins with a reduced expression for v. (Similarly, there is a "weak left order".) These weak orders are the subject of Chapter 3 in [1].

The motivation for the second partial order comes from Lie theory. Finite Coxeter groups play a decisive role in the study of Lie groups and associated geometric objects such as the "flag manifolds" G/B. (Here G is a semisimple Lie group and B is a "Borel subgroup", i.e., a maximal solvable subgroup.)[6] The manifold G/B has the structure of a cell complex with one cell for each element in the associated Coxeter group W. This defines the *Bruhat order* on W: given elements $v, w \in W$, write $v < w$ if the cell corresponding to v is contained in the closure of the cell corresponding to w. It turns out that one can give a combinatorial definition of the Bruhat order that is valid for all Coxeter groups (not only those that are associated to Lie groups). This partial order is the subject of Chapter 2 in [1].

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[4] One of the most entertaining and important sections of [1] is §4.3 on "The numbers game". Here the authors give a combinatorial method for computing generic orbits in the contragradient geometric representation and then show how to use this method to determine when a given expression $s_1 \cdots s_n$ is reduced.

[5] Since such moves do not increase $l(w)$, this gives an algorithm for deciding when a word in S represents the identity element of W.

[6] If $G = GL(n,k)$, with $k = \mathbf{R}$ or $\mathbf{C}$, then B is the subgroup of upper triangular matrices and G/B can be identified with the manifold of all "flags" of the form $F_1 \subset \cdots F_{n-1} \subset F_n = k^n$, where F_i is an i-dimensional subspace of k^n.

Kazhdan-Lusztig polynomials. To any Coxeter system (W, S) and parameter q, one can associate a "Hecke algebra". When $q = 1$, the Hecke algebra is the group algebra of W. When W is finite, the representation theory of the Hecke algebra determines the representation theory of any associated finite group of Lie type over a field of order q. In order to study the representations of the Hecke algebra, Kazhdan and Lusztig [7] defined a certain additive basis (called the "Kazhdan–Lusztig basis") for the Hecke algebra and then gave formulas relating this basis to a standard basis in terms of certain polynomials in $\mathbb{Z}[q]$. These are the mysterious "Kazhdan–Lusztig polynomials". In Chapter 5 the authors define these polynomials (without first mentioning Hecke algebras or Kazhdan-Lusztig bases) and develop methods for computing them. I prefer the discussion of this material in Humphreys' book [6]. The representation theory of Hecke algebras is discussed in Chapter 6. The discussion is soon specialized to the case of the group algebra of the symmetric group.

Growth series. Chapter 7 is entitled "Enumeration". Its first section, "Poincaré series", deals with the growth series $\sum_{w \in W} t^{l(w)}$. It turns out that this is the power series of a rational function in t. Various explicit formulas for this rational function are given in §7.1. The rest of the chapter deals with the problem of counting the number of reduced expressions for an element of the symmetric group.

The book contains a great deal more than is indicated above. For example, in §4.8 it is shown that the reduced expressions for elements of W form a regular language (i.e., they are recognized by a finite state automaton).

Each chapter concludes with a list of exercises and "Notes" which give historical and bibliographical information, as well as references for the exercises. The exercises and notes are very nice features. In particular, the exercises contain many interesting facts which cannot easily be found elsewhere.

3. Other possible approaches

I should confess that I also have written a book [5] on Coxeter groups. There are several possible approaches such a book could take. For example, one could choose to emphasize any one of the following four directions:

(1) connections with root systems and Lie theory,
(2) connections with the theory of buildings,
(3) connections with combinatorics,
(4) connections with geometric group theory.

There might well be other choices of direction. The book under review is an example of (3), while [2], [3], [5] are examples of (1), (2), (4), respectively. Each of these four viewpoints is distinct and each leads to a very different book. This is certainly the case when comparing (3) and (4). The material in [1] is almost completely disjoint from that in [5]. The same is true for the bibliographies of these books. The fact that there are such disparate approaches to this subject is a testament to the pervasiveness of Coxeter groups in mathematics.

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