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Serre's problem on projective modules, by T. Y. Lam, Springer Monographs in Mathematics, Springer, Berlin, Heidelberg, New York, 2006, xxi + 401 pp., US$99.00, ISBN 978-3-540-23317-6

For the purposes of this review we can define a projective module to be a direct summand of a free module. We will only consider commutative rings R with unit and finitely generated projective modules, which are easily seen to be direct summands of finitely generated free modules. A natural question arises immediately: Are all such modules free? In general, the answer is negative. Classical examples are given by modules over rings of algebraic integers of class number greater than 1. A trivial example is given by taking $P = A \times 0$ over a product ring $R = A \times B$. For certain classes of rings such as fields and principal ideal domains the answer is affirmative, so a better formulation would be, which rings R have the property that all finitely generated projective R–modules are free?

The most famous question of this type was raised by Serre in [19]:

Serre's Problem. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Are all finitely generated projective R–modules free?*

The answer is clearly "Yes" for $n = 0$ and $n = 1$, but for higher values of n the question proved to be extremely difficult and it took about 23 years until the problem was solved independently (and affirmatively) by Quillen [15] and Suslin [23].

Serre was led to this question by his theorem that vector bundles on an affine variety are equivalent to finitely generated projective modules over its coordinate ring. The polynomial ring $k[x_1, \ldots, x_n]$ is the coordinate ring of affine space $\mathbb{A}^n$. Since all topological vector bundles on Euclidean space $\mathbb{R}^n$ are trivial [22], it is natural to ask if the same is true algebraically. Additional evidence was later provided by a theorem of Grauert [6] which implies that analytic vector bundles on $\mathbb{C}^n$ are trivial but neither the topological nor the analytic proof generalize in any obvious way to the algebraic case.

The problem turned out to be surprisingly difficult. At first only partial results were obtained. In 1957, Seshadri [18] solved the case of 2 variables. No further significant results were found for many years until in 1974 Murthy and Towber [12] solved the case of 3 variables over an algebraically closed field. Soon after, a number of cases were obtained by Suslin and Vaserstein [26] and by M. Roitman [17]. Finally in 1976, the full solution was obtained by Quillen and Suslin.

1. Some early history

In a way it was fortunate that the problem proved so difficult, since most of the theory of projective modules over commutative rings was developed in attempts to solve Serre's problem. The first significant results were found by Serre himself [20]. He showed that a finitely generated projective module P over a polynomial ring $R = k[x_1, \ldots, x_n]$ is stably free; i.e., for some finitely generated free R-module F, $P \oplus F$ is free. Serre gave two proofs of this, one using graded rings and one

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using Grothendieck's functor K_0, which also served to introduce many commutative algebraists to the new field of algebraic K–theory. This result led to an interesting reformulation of the problem. It will clearly suffice to show that if $F = P \oplus R$ is free, so is P. It is easy to see that a non–zero element $v = (v_1, \ldots, v_n)$ of a free module $F = R^n$ generates a direct summand if and only if it is unimodular; i.e., there are elements a_i of R such that $\sum a_i v_i = 1$. The complement P will be free if and only if v is part of a free base of F and so can be transformed into $(1, 0, \ldots, 0)$ by an automorphism of F. This leads to a restatement of Serre's problem as a unimodular row problem involving only polynomials over the field k.

Unimodular Row Problem. *Let $v = (v_1, \ldots, v_n)$ be a unimodular row over the polynomial ring $R = k[x_1, \ldots, x_n]$. Is v the first row of an invertible matrix over R?*

Equivalently, is there a $g \in GL_n(R)$ such that $(v_1, \ldots, v_n)g = (1, 0, \ldots, 0)$?

Serre also showed that it would suffice to consider projective modules of rank at most n.

Theorem 1.1 (Serre [20])**.** *Let R be a commutative noetherian ring of Krull dimension d. If P is a finitely generated projective module of constant rank $r \geq d$, then $P = Q \oplus F$ where F is free and Q has rank at most d.*

It is natural to ask whether Q is unique. A very useful theorem of this sort was proved by Bass.

Theorem 1.2 (Bass [1])**.** *Let R be a commutative noetherian ring of Krull dimension d. If P and Q are finitely generated projective modules of constant rank $r > d$ and if $P \oplus F \approx Q \oplus F$ with F free and finitely generated, then $P \approx Q$.*

It follows that all projective modules of rank $> n$ over $k[x_1, \ldots, x_n]$ are free (for k a field). Bass [2] also showed that all non–finitely generated projective modules over this ring are free so that only the finitely generated case remains of interest.

These theorems can be summed up as follows: Let $P_r(R)$ be the set of isomorphism classes of finitely generated projective R–modules of rank r and define $P_r(R) \to P_{r+1}(R)$ by sending the class of P to that of $P \oplus R$. Then, for R noetherian with $\dim R = d$, we have

$$P_d(R) \twoheadrightarrow P_{d+1}(R) \xrightarrow{\approx} P_{d+2}(R) \xrightarrow{\approx} P_{d+3}(R) \xrightarrow{\approx} \ldots. \tag{1}$$

The limiting value $P_r(R)$ for $r >> 0$ can be identified with $Rk_0(R)$, the kernel of the rank map on $K_0(R)$ [3].

Needless to say, these theorems play a fundamental role in the study of projective modules over noetherian rings. More general versions were later proved by Eisenbud and Evans [5].

In the early 1970's Suslin and Vaserstein introduced many useful techniques for dealing with projective modules, including the use of symplectic methods. They also proved many results concerning the unimodular row problem, of which I will mention only the following remarkable and very useful theorem of Suslin concerning this problem.

Theorem 1.3 (Suslin's $n!$ Theorem [24])**.** *Let $(a_0, a_1, \ldots, a_n)$ be a unimodular row. If $r_0, r_1, \ldots, r_n$ are positive integers such that $n!$ divides $r_0 r_1 \ldots r_n$, then $(a_0^{r_0}, a_1^{r_1}, \ldots, a_n^{r_n})$ can be completed to an invertible matrix.*

This was proved shortly before the complete solution to Serre's problem was found. The case $n = 2$ was proved earlier in [27].

Serre [21] also discovered a very useful application concerning complete intersections. If X is an algebraic variety of dimension d in n–space defined by an ideal I, the problem is to determine when I has the minimal number of generators $n - d$ so that X is the complete intersection of $n-d$ hypersurfaces. Here is a very simple case which illustrates Serre's idea. Suppose I is an ideal of height 2 in the polynomial ring $R = k[x, y]$. We want to know if I can be generated by 2 elements. A necessary condition is that this should hold locally; i.e., for each maximal ideal $\mathfrak{m}$ of k, $I_{\mathfrak{m}}$ should have 2 generators. Assuming this, one can calculate with the Koszul complex that $\mathrm{Ext}^1(I_{\mathfrak{m}}, R_{\mathfrak{m}}) \approx R_{\mathfrak{m}}/I_{\mathfrak{m}}$, and one can conclude easily that $\mathrm{Ext}^1(I, R) \approx R/I$. The generator corresponds to an extension $0 \to R \to P \to I \to 0$, and Serre shows that P is a projective R–module. Since $\operatorname{rk} P = 2$, Seshadri's theorem shows that $P \approx R^2$. Since P maps onto I, I has 2 generators as required. This approach has played a very important role in the subsequent study of the complete intersection problem.

2. The solution

Quillen's solution of the problem is based on a Local–Global principle. Instead of proving directly that a module is free, he considers a relative version. A module M over a polynomial ring $R[X]$ is called extended if $M \approx R[X] \otimes_R N$ for some R–module N. (Of course $N = M/XM$). If $\mathfrak{m}$ is a maximal ideal of R, write $M_{\mathfrak{m}}$ for the localization $M_{R-\mathfrak{m}}$. The following is Quillen's basic result.

Theorem 2.1 (Quillen's Patching Theorem [15])**.** *Let M be a finitely presented module over $R[X]$ where R is a commutative ring. If $M_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R, then M is extended from R.*

Horrocks [8] had previously shown that if R is a local noetherian ring, P is a finitely generated projective module over $R[X]$, and $P[f(X)^{-1}]$ is free for a monic polynomial $f(X)$, then P is free. Quillen's theorem shows that this can be globalized; namely, if R is not assumed local and $P[f(X)^{-1}]$ is extended from a projective R–module, then so is P. Now if P is a finitely generated projective module over $k[X_1, \dots, X_n]$ where k is a field, invert all monic polynomials in X_1 getting a projective module over $k(X_1)[X_2 \dots, X_n]$ which will be free by induction on n. It follows that $P[f(X)^{-1}]$ is free for some monic polynomial $f(X_1)$ so P is extended from a projective $k[X_2, \dots, X_n]$–module and is therefore free by the induction hypothesis once again. Quillen observes that the same argument will work for projective modules over $R[X_1, \dots, X_n]$ if R is a principal ideal ring.

At the same time as Quillen's work, Suslin found a completely different proof. These proofs were quickly followed by elementary proofs due to Suslin and Vaserstein. Vaserstein even gave a complete proof in 8 lines! These proofs are stated in terms of the unimodular row problem.

Suslin's elementary proof actually proves a stronger result, namely that the element g of $GL_n(R)$ such that $(v_1, \dots, v_n)g = (1, 0, \dots, 0)$ can be taken to be a composition of elementary transformations of the form $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_i + ax_j, \dots, x_n)$ obtained by adding a multiple of x_j to x_i for some $j \neq i$ (as in the familiar construction of the Euclidean algorithm). These generate the subgroup $E_n(R)$ of $GL_n(R)$. The theorem actually applies to any noetherian ring.

Theorem 2.2 (Suslin). *Let A be a commutative noetherian ring and let $R = A[x_1, \dots, x_m]$. If $n \geq \max(3, 2 + \dim A)$, then any unimodular row $(v_1, \dots, v_n)$ over R can be reduced to $(1, 0, \dots, 0)$ by a series of elementary transformations.*

Note that the bound on n is independent of m. The solution to Serre's problem follows immediately, since if A is a field, the bound is $n \geq 3$ and unimodular rows of length 2 can be completed trivially.

3. Subsequent work

Once Serre's problem was solved there were naturally a number of related problems to which the methods were applied. Perhaps the best known of these problems is the following. I have phrased this conjecture as a question to avoid confusing the casual reader. A ring is called regular if it is noetherian and its localizations at prime ideals are regular local rings.

Bass–Quillen Conjecture. *If R is a regular ring, is every finitely generated projective module over $R[x_1, \dots, x_n]$ extended from R?*

This was shown to be true for $\dim R \leq 2$ by Quillen [15] and by Suslin [23] (independently) using a theorem of Murthy and Horrocks [11] for the local case. Rao [16] has done the case $\dim R = 3$ with $\frac{1}{6} \in R$. For higher dimensions the following is the best result obtained so far.

Theorem 3.1 (Lindel [10]–Popescu [14]). *If R is a regular ring containing a field, then every finitely generated projective module over $R[x_1, \dots, x_n]$ is extended from R.*

The proof also applies to unramified regular local rings, but the general case is still open.

In another direction, the problem has been extended to the quadratic case. We assume that the characteristic is not 2 for simplicity. By a quadratic module I will mean a finitely generated projective module with a non–singular quadratic form.

Quadratic Analogue of Serre's Problem. *Let k be a field of characteristic not 2. Is every quadratic module over $k[x_1, \dots, x_n]$ extended from k?*

In this case the results are not so easy to describe, since Parimala [13] has shown that in general the answer is "No", but there are cases in which the answer is affirmative.

There has also been quite a bit of work done on extending the results on Serre's problem to rings more general than polynomial rings. A theorem of this sort has been proved for discrete Hodge algebras by Vorst [29], and J. Gubeladze [7] proved a conjecture of Anderson which extends the solution of Serre's problem to a large class of monoid rings.

In quite a different direction, Suslin [25] proved a K_1–analogue of Serre's problem for the special linear group. As observed above, $Rk_0(R)$ is the colimit of the sets $P_r(R)$. Similarly, $SK_1(R)$, the kernel of the determinant map on $K_1(R)$, is the colimit of the sets $SL_r(R)/E_r(R)$ [3]. This suggests an analogy between these two sequences of sets. Suslin [25] proved the surprising result that for a commutative ring R, $E_n(R)$ is a normal subgroup of $SL_n(R)$ for $n \geq 3$ so that $SL_n(R)/E_n(R)$ is a group. By analogy with (1) we can consider the sequence formed by the $SL_n(R)/E_n(R)$:

$$SL_1(R)/E_1(R) \to SL_2(R)/E_2(R) \to SL_3(R)/E_3(R) \to \dots \tag{2}$$

with limit $SK_1(R)$. Bass [1], [3] conjectured and partially proved that if R is noetherian and $d = \max(3, 2 + \dim R)$, then we have

$$(3) \qquad SL_{d-1}(R)/E_{d-1}(R) \twoheadrightarrow SL_d(R)/E_d(R) \xrightarrow{\approx} SL_{d+1}(R)/E_{d+1}(R) \xrightarrow{\approx} \dots .$$

The full result was proved by Vaserstein[28]. This was extended by Suslin as follows.

Theorem 3.2 (Suslin [25])**.** *Let A be a commutative noetherian ring and let $R = A[x_1, \dots, x_m]$. If $d = \max(3, 2 + \dim R)$, then (3) holds for R.*

The ontoness follows from Theorem 2.2, while the injectivity is proved by reducing to Vaserstein's theorem. As in Theorem 2.2, the bound d is independent of m. In particular, if A is a field, then $SL_r(R) = E_r(R)$ for $r \geq 3$. This result can be considered a K_1–analogue of Serre's problem (for $r \geq 3$) since it shows that $SL_r(R)/E_r(R)$ is trivial while Serre's problem states that $P_r(R)$ is trivial for $R = k[x_1, \dots, x_m]$ where k is a field.

Suslin also proved the following Local–Global theorem, which is an analogue of Quillen's Patching Theorem.

Theorem 3.3 (Suslin [25])**.** *Let $\alpha(X) \in SL_n(R[X])$ satisfy $\alpha(0) = 1$. Assume that $n \geq 3$. If the image $\alpha(X)_{\mathfrak{m}}$ of $\alpha(X)$ in $SL_n(R_{\mathfrak{m}}[X])$ lies in $E_n(R_{\mathfrak{m}}[X])$ for all maximal ideals $\mathfrak{m}$ of R, then $\alpha(X)$ lies in $E_n(R[X])$.*

4. The book

I hope the preceding outline will inspire the reader to learn more about Serre's problem and related problems. If so, he or she can do no better than to consult Lam's book. Lam has done a magnificent job of organizing the material and presenting complete proofs of all the results directly connected with Serre's problem. He also gives a very thorough discussion of recent work and of related topics not directly relevant to Serre's problem. The references are complete and make the book a very valuable reference even for experts in the field. It is certainly the first place one should look for anything related to Serre's problem.

The book begins with a detailed account of the work done prior to the solution of Serre's problem. This is followed by separate chapters devoted to the elementary proofs of Suslin and Vaserstein (including Suslin's $n!$ theorem), Horrock's theorem, and Quillen's Patching Theorem. Everything directly connected to Serre's problem is proved in detail. The remaining chapters cover Suslin's work on the special linear group, results on the quadratic case, and applications to the complete intersection problem. A final chapter summarizes work done since the solution of Serre's problem. Here most proofs are omitted but complete references to the original papers are given. I found this chapter extremely useful in searching for results related to Serre's problem, projective modules, etc. This chapter also contains references to various applications of Serre's problem to other fields of mathematics.

This book is a greatly expanded version of Lam's 1978 volume in the Springer Lecture Notes, but even readers thoroughly familiar with these notes will find much of interest in the present book. It will be very useful to students wishing to learn about projective modules and also to those familiar with this subject who wish to look up the statement of a needed result or to look for a reference to recent work. This is definitely a book that anyone even remotely interested in projective modules should have on his or her shelf!

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