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Soliton equations and their algebro-geometric solutions. Vol. I. $(1+1)$*-dimensional continuous models*, by Fritz Gesztesy and Helge Holden, Cambridge Studies in Advanced Mathematics, 79, Cambridge University Press, Cambridge, 2003, xii+505 pp., US$117.00, ISBN 0-521-75307-4

"Integrable Systems" has become a field of mathematics in relatively recent times (triggering frequent additions to the Mathematics Subject Classification), one with the odd distinction that a definition of integrability is still lacking. The reason for this elusiveness and also for much of the fast hold the field is enjoying on (applied) mathematics and physics is that nonlinear PDEs typically don't have "exact" solutions (elementary or special functions, e.g., elliptic, hypergeometric) and of course the solution space is infinite-dimensional. Classes of "integrable" PDEs with special properties were identified, beginning in the 1960s; there is still no way to predict whether your equation will be integrable, but those that are seem to enjoy an extraordinarily beautiful mathematical structure. Uncovering such structure in the attempt at finding defining properties of, and tests for, integrability brought to light new mathematical constructions which span classical to futuristic interdisciplinary areas. Not surprisingly, then, this part of the mathematical sciences is one that has seen the largest growth in the number of publications (the book under review comes with a reference list which, though complete, is limited to the pertinent topics and contains 800+ items), and texts are now available to give the field some rooting and the newcomer some means to handle the necessary tools in a comprehensive way. Most of the texts that succeed have chosen one specific viewpoint and technique; the present one is of this kind and quite unique in its choice.

The reason for this review is that this is the 'first' text to treat algebro-geometric integration married in a deep way to spectral theory and classical analysis: it is this choice that gives this area of the elusive integrability field a definitive completeness (which does not mean there aren't plenty of open problems in the book). This review is not as timely as it should have been (my regrets!), but there still seems to be a reason for it: this book is the first of three, the second is near completion, and I will try to provide scope for the whole series (Volume I treats the "hyperelliptic" case, which in applied math translates into holonomic D-modules or the evolution of one spatial dimension, $(1+1)$ in the title; Volume II is planned to address the discrete case or (differential-)difference equations; and Volume III is to include the non-hyperelliptic, subholonomic case, $(2+1)$ for 2 spatial dimensions). The 2006 AMS Leroy P. Steele Prize for a Seminal Contribution to Research was awarded to the authors of the paper [GGKM]; the citation reads: "This is a fundamental paper in the theory of solitons, inverse scattering transforms, and nonlinear completely integrable systems. Before it, there was no general theory for the exact solution of any important class of nonlinear differential equations. . . . In applications of mathematics, solitons and their descendants (kinks, anti-kinks, instantons, and breathers) have entered and changed such diverse fields as nonlinear optics, plasma physics, and ocean, atmospheric, and planetary sciences. Nonlinearity has undergone a revolution: from a nuisance to be eliminated, to a new tool to be exploited." Indeed,

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the Princeton lab where the word *soliton* was coined in the mid-1960s is where our journey begins (for a more extensive account of the history of this subject, see the wonderful article [P] in this *Bulletin*).

The word soliton is used loosely to refer to (traveling) solitary waves that regain their individual shape and velocity after non-linear superposition. This phenomenon was identified through computer simulation—the work of C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, C.H. Su, and N.J. Zabusky. The nonlinear evolution equation observed was the Korteweg-de Vries equation ([KDV], though it appeared earlier [B]),

$$u_t - 6uu_x + u_{xxx} = 0.$$

Exact solutions were found in the 19th century under the one-wave ansatz: $u(x,t) = u(x-ct)$ because of the minor miracle of repeated integration (with arbitrary constants a, b) into:

$$\frac{(u')^2}{2} = u^3 + c\frac{u^2}{2} - au + b,$$

the equation of the Weierstrass $\wp$ function, up to linear transformation $u = 2\wp +$ const., or its singular trigonometric and exponential limits for particular values of a, b. The Princeton group paired two one-wave solutions and observed that a numerical solution, after looking all lumpy for a while, asymptotically regained exactly the 2-peak shape with a shift in phase. This suggested the "hidden symmetry" of more conservation laws than the obvious energy. The next stage of the journey sees extraordinary geographic breadth and insight, notably of L.D. Faddeev, P.D. Lax, S.P. Novikov, Mikio Sato, who were among the leaders of the mathematical teams at the major centers of activity, familiarly referred to as the "St. Petersburg school", the "New York school", the "Moscow school", the "Kyoto school" (though many others made fundamental contributions, including applied mathematicians and physicists). In a way, the end of the journey was reached with the discovery of the structure shared by the integrable equations, to wit, the abelian sum all over again, as in the case of the simple pendulum. Algebraic tori, the venue of complete integrability, and action-angle variables were responsible for the stability of the solitons, one of the new powerful features being that the tori were infinite dimensional. Sato's Grassmannian and its determinant bundle became a "universal" setting where moduli spaces of curves (or maps or bundles) of arbitrary genus could be mapped and made to interact; this suited string theory, conformal blocks, statistical mechanics (lattice theory), stochastic processes (Brownian motion, random matrices), and more.

There weren't, however, soliton texts based on spectral theory that provided the equipment and the specificity to encompass all the "algebro-geometric solutions". The algebraic nature of the spectral data was developed in [GD], but the algebro-geometric class appeared in an unrelated way, as we see next.

What is a spectral curve? A spectral curve, like most interesting objects, has many manifestations, but for the sake of brevity we sketch the more analytic and algebraic ones. If the operator L has an honest spectral class and depends on a parameter t according to a Lax pair $\partial_t L = [B, L]$ for an operator B that can be 'integrated' on a group of transformations by a unitary operator U: $B = U_t U^{-1}$, then the Lax pair is the infinitesimal (Lie-algebra) action of the flow: $L(t) = UL(0)U^{-1}$, which preserves the spectrum of L. At least formally, this is an explanation for the conserved quantities (the eigenvalues of L), and this approach was developed rigorously for certain classes of potentials (e.g., periodic or rapidly vanishing on the line). That there should be a curve of such spectral values is a little bit more mysterious. Switch buzzwords to "non-linear Fourier transform". With a KdV solution $u(x,t)$ in mind, take L to be the Schrödinger operator $L = -\frac{d^2}{dx^2} - u(x,t)$. Writing henceforth $\partial = \frac{\partial}{\partial x}$ instead of $\frac{d}{dx}$, one can give the Lax-pair deformation above by $B = -4\partial^3 - 3(u\partial + \partial u)$. A point λ in the spectrum corresponds to an eigenfunction, $L\psi = \lambda\psi$, whose asymptotic behavior (don't ask yet!) is $e^{x\lambda^{1/2}}$. (Had L been a partial differential operator in two variables, x and y, say, one might have looked for an eigensurface parametrized by λ, μ, say, but that's material for another three books.) The geometric data corresponding to the eigenfunction are of a deeply different nature, depending on the class of potentials studied; this is why we put off the question "asymptotic *where*?" For the rapidly vanishing class, it is the asymptotics as $x \to \pm\infty$ on the real line that provide "scattering data", the surprise of their simple dependence on t (polynomial in fact), and trace formulas to recover the potential $u(x,t)$, yielding (if all the auspicious assumptions hold) an *exact* KdV solution. The Riemann-Hilbert problem, namely connecting eigenfunctions across the real $\lambda^{1/2}$ axis, completes the geometry, and again under favorable circumstances there is a global eigenfunction on a Riemann surface (of possibly infinite genus) that covers the $\lambda^{1/2}$-plane.

But it is the periodic case that we address here. In that case the geometric data that allow us to recover the potential are encoded in the asymptotics of the eigenfunction in λ. How can that be?

It is perhaps best, rather than conjuring up the magic words "Fourier-Mukai transform", to make a 'formal' excursion, especially since the underlying representation-theoretic aspect is not covered in this book. It was the insight of Sato that unified the analytic and algebraic, in fact "all", solutions into the universal Grassmann manifold [S]. He defined the spectral curve as the spectrum (in the sense of algebraic geometry—nothing technical, simply points corresponding to the maximal ideals, *de facto* their complex coordinates) of the ring $\mathcal{D}$ of (formal) differential operators that commute with the (formal) pseudo-differential operator $L^{1/2}$, $\mathcal{D} = \{\sum_{j=0}^{n} u_j(x)\partial^j,\ u_j$ a formal power series in $x\} \subset \mathcal{P} = \{\sum_{-\infty}^{n} u_j(x)\partial^j\}$, with operations: $\partial \circ u = u\partial + u'$, $\partial^{-1} \circ u = u\partial^{-1} - u'\partial^{-2} + u''\partial^{-3} - \dots$. Noting that the automorphisms of $\mathcal{D}$ are generated by a change of variable and conjugation by a function, we normalize any $L \in \mathcal{D}$ of order n:

$$L = \partial^n + u_{n-2}(x)\partial^{n-2} + \ldots + u_0(x),$$

and there is a unique n-th root: $\mathcal{L} = \partial + u_{-1}\partial^{-2} + \ldots \in \mathcal{P}$ which plays the role of a local parameter on the curve at a smooth point. The KdV hierarchy can be viewed as the isospectral deformations of the ring that centralizes the Schrödinger L, which are all defined by *compatible* Lax pairs:

$$\partial_{t_n}\mathcal{L} = [(\mathcal{L}^n)_+, \mathcal{L}].$$

It was then natural to define the more general KP hierarchy. Introduce parameters $\underline{t} = (t_1 = x, t_2, t_3, \ldots)$, and the deformations

$$\partial_{t_j}\mathcal{L} = [(\mathcal{L}^j)_+, \mathcal{L}],$$

where $(\ )_+$ is projection $\mathcal{P} \to \mathcal{D}$, are then a set of PDEs on $u_i(\underline{t})$, which turn out to be commuting Hamiltonian flows. The first non-trivial PDE, for $x = t_1,\ y = t_2,\ t = t_3$, is another equation (KP) that had been encountered in physics, modeled first by Kadomtsev and Petviashvili; moreover, the KP equation reduces to both the KdV and the Boussinesq equations, respectively, when the solution happens to be "stationary" $(\mathcal{L}^j \in \mathcal{D}(\Rightarrow \partial_{t_j}\mathcal{L} = 0))$ with respect to t_2, t_3, respectively:

$$\tfrac{3}{4}u_{yy} = (u_t - \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x)_x \text{ (KP)}$$

$$\xrightarrow{\partial_s \equiv 0} \quad 4u_t - u_{xxx} - 6uu_x = 0 \text{ (KdV)}$$

$$\xrightarrow{\partial_t \equiv 0} \quad 3u_{yy} + u_{xxx} + 6uu_x = 0 \text{ (Boussinesq).}$$

At first sight, these centralizer rings do not have much structure: a powerful result of Sato (proved earlier by I.M. Krichever in the algebro-geometric context) says that any commutative subring of $\mathcal{P}$ can be simultaneously conjugated into a subring of $\mathcal{P}_0 = \{\sum_{-\infty}^{n} c_j \partial^j\}$, constant-coefficient operators. In a short paper published as early as 1905, I. Schur proved that the centralizer of a differential operator in $\mathcal{D}$ is in fact simply $\{\sum_{-\infty}^{N} c_j \mathcal{L}^j \in \mathcal{P},\ c_j \in \mathbb{C}\} \cap \mathcal{D}$. This ring carries the geometric structure; it no longer is necessarily isomorphic to a polynomial ring $\mathbb{C}[x]$ (though it is for most operators). At the same time, being a ring of transcendence degree one over $\mathbb{C}$, it is the ring of an affine algebraic curve.

We've reached harbor: another strikingly early result of J.L. Burchnall and T.W. Chaundy (1923), elementary in nature, says that any two commutative ordinary differential operators satisfy (identically in the variable x) a polynomial equation: thus, the "spectral curve"!

From this point of view, rather close to that of the nineteenth century, results have a double nature in algebra and spectral theory. More detail when we discuss the present book's content. Here we complete our bird's-eye view of the field by using the Fourier duality between x and z to transition back to the z variable.

The 'trivial' situation of a constant-coefficients operator (equivalently, the centralizer is spanned by all the powers of $\mathcal{L}$) is coded into the general one by the "Sato operator" S: $\mathcal{L} = S\partial S^{-1}$, so $\psi = Se^{xz}$ is an eigenfunction ("Baker—or Baker-Akhiezer—function", for historical reasons) of $\mathcal{L}$ with eigenvalue z, which turns differentiation into multiplication, $\partial^{-1} \leftrightarrow z$. This enables us to work with an (infinite-dimensional) Grassmann manifold, the subspaces W of a Hilbert space $H = \mathrm{L}^2(S^1, \mathbb{C})$ which are "comparable" to $H_+ = \{\sum_0^{\infty} c_j z^j \in H\}$ (in Sato's formal version the condition is finite-dimensionality of certain quotient spaces). We denote by $\mathrm{Gr}^{\emptyset}$ the big cell, where by definition the projection $\pi_+ : H \to H_+$ is an isomorphism.

There is a 1:1 correspondence between elements of $\mathrm{Gr}^{\emptyset}$ and $\mathcal{D}$-submodules $\mathcal{I}$ of $\mathcal{P}$ that are cyclic, $\mathcal{I} = \mathcal{D}S$, equivalently split $\mathcal{P} = \mathcal{I} \oplus \mathcal{P}^{(-1)}$, through the assignments:

$$\mathcal{I} \mapsto W = S^{-1}\mathcal{P}^{(0)}_{\text{const}} = \{v \in \mathcal{P}_{\text{const}} :\ \mathcal{I}v \subset \mathcal{P}^{(0)}_{\text{const}}\},$$
$$W \mapsto \mathcal{I} = \{A \in \mathcal{P} :\ AW \subset \mathcal{P}^{(0)}_{\text{const}}\},$$

where $\mathcal{P}_{\text{const}}$, the constant-coefficient operators, are equipped with the filtration by order $\mathcal{P}^{(j)}_{\text{const}}$. Now each element of $\mathrm{Gr}^{\emptyset}$ corresponds uniquely to its Baker function, the inverse image of 1 under π_+. To understand the geometric interpretation of the KP flows on the Grassmannian, Sato used "Plücker coordinates" (suitable minors of infinite-size matrices), devised a generating function for them (the seemingly omnipresent "tau function"), and proved his crowning result: the KP hierarchy is nothing but the Plücker equations, a generalization of the classical quadratic equations that cut out the image of the Plücker embedding of $\mathrm{Gr}(k, n)$ in projective space $\mathbb{P}^{\binom{n}{k}-1}$. The hierarchy is linear on the Grassmannian in the sense that the subspaces corresponding to $S(t_1, t_2, ...)$ are $W(t) = e^{t_1 z + t_2 z^2 + \cdots} W$. As a passing nod to the vast contribution made by representation theorists to integrable equations and a foray into Volume II, where the Toda Lattice,

$$w_{i,xx} = \exp(w_{i-1} - w_i) - \exp(w_i - w_{i+1}), \quad i = 1, \cdots, n,$$
$$\sum_{i=1}^{n} w_i = 0, \quad w_0 = w_n, \quad w_{n+1} = w_1,$$

will take center stage, we connect the two by mentioning that Toda likewise linearizes on a vector Grassmannian under the action of the loop group of $\mathrm{SL}(n, \mathbb{C})$. The flow on the Grassmannian is now generated by the positive part of the Heisenberg algebra $H_{\underline{n}}$ as follows. A map $f : S^1 \to \mathrm{Gl}_n(\mathbb{C})$ is written, by an infinite-dimensional version of the Birkhoff decomposition, as a product $f = f^+ \lambda f^-$ of a homomorphism λ and maps $f^{\pm}$ that extend holomorphically to the interior and exterior of $S^1 \subset \mathbb{C} \cup \{\infty\} = S^2$, respectively. The action ("dressing transformation") is defined by factoring a product $f^+ f^-$ in the opposite order. The space of solutions is the orbit of a group action, which in turn lends itself to integration by symplectic reduction. But this is for another review.

This then is the story of the soliton, if you will, told by someone biased towards algebraic geometry: the two algebraic varieties that were classically best understood, namely abelian varieties (more specifically Jacobians of Riemann surfaces) and Grassmannians, are the milieu of integrable hierarchies (you may think you are not seeing the Jacobian inside the Grassmannian, but it is encoded via the "Krichever map", which associates to a line bundle E the subspace W of its local sections near $z = \infty$, most sketchily speaking). The puzzle of "unreasonable effectiveness" has been clarified, if not explained; why is it that the PDEs of 'nature', when expanded in what some view as an analog of the "normal form" of Hamiltonian systems, have first approximation which is an integrable equation in the sense of belonging to a linear hierarchy? Perhaps because "a lot of what we see comes from the glasses we put on" [H]. Be that as it may, soliton equations have brought about, and together, a truly unreasonable number of things.

The algebraic Ansatz. As a *segue* into the book at hand, let's pause to see how, within the context of an infinite-dimensional Grassmannian, the finite-dimensional Jacobian still might host the whole hierarchy. This is for the special solutions (initial conditions) which are "algebro geometric". Again, only one (basically because of the one-dimensionality of the spectral variety) relation suffices: let $K_j = (\mathcal{L}^j)_+$ and say that a KP solution is stationary if a non-trival combination $\sum_1^N c_j \mathcal{L}^j \in \mathcal{D}$; i.e., the corresponding time operator $\sum_1^N c_j K_j$ acts trivially. A technical proviso is important: the greatest common divisor of the orders of the operators giving rise to stationary flows has to be 1 (this is commonly referred to as the "rank 1" condition).

It turns out (again the exposition here leaves out lots of technical provisos) that this is equivalent to an exquisitely algebraic question asked by Burchnall and Chaundy: in the slightly less formal ring $\mathcal{D} = \{\sum_{j=0}^n u_j(x)\partial^j,\ u_j$ analytic near $x = 0\}$, which L's have centralizer $\mathcal{C}_{\mathcal{D}}(L)$ larger than a polynomial ring $\mathbb{C}[L_1],\ L_1 \in \mathcal{D}$? The rank-one condition says that the orders of the operators in the centralizer should be coprime. Ince's classical ODE result gives a feeling for the depth of the problem: the Lamé operator $L = -\partial^2 + a(a+1)\wp(x-x_0)$ with real, smooth potential is finite-gap iff $a \in \mathbb{Z}$ (if a is positive, the spectral curve has genus a). To give a taste for the techniques of the authors of the book under review, we pursue this example for a moment in the rather unique treatment of the first author with R. Weikard: they give a necessary and sufficient condition for an operator $L = d^2/dx^2 + q(x)$ (where $x \in \mathbb{C}$, q is meromorphic doubly periodic) to be finite-gap by extending a theorem of Picard; the condition is that q be what they term a "Picard potential", i.e., admit a fundamental system of meromorphic solutions of $\psi'' + q\psi = E\psi$ for any $E \in \mathbb{C}$.

The book at hand takes this analytic approach to algebro-geometric solutions of soliton equations. Its nature as a consequence is rather unique in the literature and it is twofold. On one hand, it is again purely algebraic, because to stay within the finite genus context, one starts with the infinite sequence of conserved quantities of KdV (for example), which have come to be viewed as commuting Hamiltonians on an infinite-dimensional Poisson manifold (but that's for another review, remember?). But it can be phrased in analytic terms as well, the algebraic objects being interwoven in asymptotic expansions (not their truncation!) of Green's functions, kernels, and Fredholm determinants associated to the spectrum, rather than holomorphic differentials on the spectral curve and linear flows on the Jacobian.

KdV is defined by recursive time evolutions:

$$u_{t_n} = [P_{2n+1}, L] = 2f_{n+1,x},$$

where $f_0 = 1,\ f_{l,x} = -\frac{1}{4}f_{l-1,xxx} + uf_{l-1,x} + \frac{1}{2}u_x f_{l-1}$ are differential polynomials in the smooth, complex-valued function u. The analytic origin of these polynomials is seen by expanding (in the spectral parameter) the Green's function—the kernel of the resolvent $(L-\lambda)^{-1}$, appeared previously in work of R. Schimming. The solutions are algebro-geometric if and only if a constant-coefficient combination of a finite number of time evolutions is zero, and the authors transparently interpret this Ansatz (appearing previously in work of S.J. Alber) in terms of the Burchnall-Chaundy theory, which they call "stationary" KdV. Joint eigenfunctions have poles on a divisor of the spectral curve that parametrizes isospectral deformations and zeros on divisors whose coordinates satisfy the "Dubrovin equations"; trace formulas are available to recover the coefficients of the operators in terms of the theta function of the curve by integrating the Dubrovin equations, both as dependent on x and on the time hierarchy t_n. Again, the analytic way to see the auxiliary divisor (the zeros of the joint eigenfunction, the Baker function redux) is available in the special case when the potential $u(x)$ is periodic; poles and zeros are then natural normalizations of spectral data, namely the spectra for the Neumann and Dirichlet boundary conditions. In fact, I have seen only in this book the most general interpolating boundary conditions for a (normalized) eigenfunction $g(x)$ of L (the t-dependence is omitted in this notation, since it is isospectral when KdV is satisfied). The condition is: $g'(x) + \ell g(x) = 0,\ \ell \in \mathbb{R}$, the case of $\ell = \infty$ and $\ell = 0$ defining respectively the Dirichlet and Neumann spectra. The KdV dynamics is reduced to a system of ordinary differential equations for the points $\lambda_0^\ell, \dots, \lambda_g^\ell$ of the spectrum. Trace formulas give: $2\ell^2 - u = \sum_{i=0}^{2g} e_i - 2\sum_{i=0}^{g} \lambda_i^\ell$, where $e_0, \dots, e_{2g}$ are the simple points of the (anti-)periodic spectrum, a finite number in the algebraic ansatz. Theta-function theory settles the 'direct' spectral problem: $u(x) = \sum_{i=0}^{2g} e_i - 2\sum_{i=1}^{g} r_i - 2\partial^2 \log \vartheta(K + A(\sum_{i=0}^{g} \lambda_i^\ell) + Ux)$, with r_i determined by meromorphic differentials, A the Abel map, U a suitable number, and the Riemann theta function:

$$\vartheta(z) := \sum_{m\in\mathbb{Z}^g} \exp\left(\frac{1}{2}m\Omega^t m + z^t m\right),$$

with normalized period matrix $[2\pi i I, \Omega]$ associated to the hyperelliptic curve of genus g branched at $e_0, \ldots, e_{2g}, \infty$. More general curves provide algebro-geometric solutions to the KP hierarchy,

$$u(\underline{t}) = -2\frac{\partial^2}{\partial t_1^2}\log\vartheta\left(\sum_{i\geq 1} t_i U_i + U_0\right) + \text{const},$$

where $U_i \in \mathbb{C}^g$ are suitable vectors and the constant depends on the curve.

Equivalently, the KdV hierarchy is obtained as the zero-curvature condition

$$\partial_{t_n} U - \partial_x V_{n+1} + [U, V_{n+1}] = 0, \quad \text{where} \quad U = \begin{bmatrix} 0 & 1 \\ -z + u(x,\underline{t}) & 0 \end{bmatrix}$$

and V_{n+1} is a suitable matrix depending on $x, \underline{t}$ and polynomially on z. While this is just the $(d \times d)$-matrix version of a d-th order linear differential equation, it is a powerful formalism when generalized (replacing z by a rational function changes the spectral curve; imposing conditions on the entries of the matrix or, more daringly, replacing differentiation by integration, a non-local operator, changes the equations) which encodes all the equations treated in this book.

Arts and crafts. The book is deftly crafted, consisting of five main chapters, each of which takes up in turn, and in parallel, one main class of integrable equations. This is an excellent choice for at least two reasons, theoretical and pedagogical. Theoretically, the geometric picture progresses from the earliest example (KdV), whose spectral curves are hyperelliptic with a branchpoint at infinity (the point with local parameter z^{-1}, z^2 in the spectrum), to the curves of sine-Gordon and modified KdV, then the more general hierarchy called AKNS from the initials of the devisers (M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur), where infinity need not be ramified, on to the massive Thirring model (the flows on the Jacobian have directions differing in nature from the previous in that the hyperelliptic conjugation does not act on their ensemble), and the most recently studied Camassa-Holm equation, the one which is to date the more geometrically mysterious, admitting solutions ("peakons") with non-meromorphic behavior. The spectral curve is still hyperelliptic, solutions are expressed by the theta function; but in deriving the Dubrovin equations, the authors note that the divisor map does not linearize the hierarchy; the non-linear behavior is interpreted in terms of a constraint, and this in turn is interpreted as non-linear motion on the generalized Jacobian.

The parallel treatment makes for some repetition, but one that is pedagogically efficient: the cases are different enough that the thorough development of the formulas and the techniques will result in sound training. This thoroughness is part of the reason why the authors do not provide exercises (as well as the fact that non-trivial exercises in the theory would amount to research projects).

In ten appendices, the authors cover economically the foundations of what areas are needed to fully use the techniques, e.g., Riemann surfaces and theta functions, Lagrange Interpolation, Herglotz and Weyl-Titchmarsh functions of spectral theory. This provides a complete, self-contained, user-friendly resource.

The least pleasant obligation of a reviewer is to ask what is missing from the book. The authors volunteer a list in the Introduction (and the overview given above perhaps provides a hint or two). In addition, there isn't information on 'future directions' (e.g., Frobenius manifolds and isomonodromy in Hurwitz spaces, where the tau function appears again, conformal blocks, string theory, Gromov-Witten invariants); there is no mention of alternative models (e.g., dispersionless hierarchies, Whitham equations, non-canonical normal form and perturbation of finite and continuous Hamiltonian systems, calculus of variations on infinite-dimensional manifolds) or striking applications to classical and contemporary differential geometry (e.g., differential invariants, harmonic maps, minimal surfaces, the Willmore conjecture), but if I were to name just one aspect of these equations that is left out of the book, I would say it is the Hamiltonian (symplectic, or Poisson if you prefer, with attendant buzzwords R matrices and quantum groups) structure. It is present in a very significant way in all of the formulas in this book, but the authors chose not to analyze the dependence of the solutions on the moduli of the spectral curve. To me, that dependence provides the second reason for the pervasiveness of algebraic geometry in physics: the abelian sum takes place on one integral manifold, then the heat connection allows you to walk through the phase space and completes the dual nature of the theta function, linking it with modular forms. Foundations on that aspect (written by some of its originators) can be found in [DKN] and [FT], different from each other in their viewpoints on symmetries: constraint equations in the former, reduction of invariants in the latter.

While these 'omissions' are deliberate, the book more than compensates by filling significant gaps, mainly in the 'coda' (the "Notes" sections) that accompanies each chapter, including most of the appendices. The "Notes" provide a comprehensive history of the equations, including the nineteenth century; alternative spectral formulas (e.g. scalar operator to matrix); real-valuedness; periodicity constraints; topological properties; and more.

A combination of hard analysis and a lucid geometric justification of the mystery of integrability in the algebro-geometric case, including classical intuitions (for one example, the Darboux-Bäcklund transformation of the KdV potential,

$$u(x,t) = -2\wp(x-ct) \longrightarrow -2\wp(x-ct-e)$$

is nothing but conjugation by $(\partial - \frac{\psi'}{\psi})$ in the ring $\mathcal{P}$, ψ the Baker function at the point $P = (\wp(e), \frac{1}{2}\wp'(e))$ of the spectrum), providing unity to the 800+ bibliography, makes this book a very valuable resource.

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