BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 1, January 2009, Pages 143–150
S 0273-0979(08)01208-1
Article electronically published on September 15, 2008

Double affine Hecke algebras, by Ivan Cherednik, London Mathematical Society, Lecture Note Series, 319, xii+434 pp., US$79.00, ISBN 13: 978-0-521-609180

The quantum Yang-Baxter equation, which we will explicitly describe in a moment, surfaced in independent work of C.N. Yang and R.J. Baxter as a profound structural equation behind certain physical models. Soon it was realized that the quantum Yang-Baxter equation has numerous important applications in mathematics and physics. For example, solutions of quantum Yang-Baxter equations are used to produce integrable systems as well as topological invariants of knots, links, and 3-manifolds. These insights have led to a host of new research, aiming at the development of mathematical tools to produce nontrivial solutions of quantum Yang-Baxter equations.

The theory of quantum groups, for instance, initiated by Drinfel'd and Jimbo, leads to universal solutions of the quantum Yang-Baxter equation called universal R-matrices. A significant ramification of this line of investigation is the rigorous mathematical construction of Witten's topological 3-manifold invariants, an achievement which was highlighted by the Fields Medals of Drinfel'd, Jones and Witten in 1990.

A second key development is mainly due to Ivan Cherednik. This theory produces universal solutions of generalized quantum Yang-Baxter equations. The role of quantum groups is taken over by Hecke algebras. The book under review provides an extensive introduction to Cherednik's theory and to its numerous applications.

Quantum Yang-Baxter equation. Let V be a complex vector space. An endomorphism $R(x)$ of $V \otimes V$, depending meromorphically on $x \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, is said to satisfy the quantum Yang-Baxter equation (with spectral parameters) if

$$R_{1,2}(x)R_{1,3}(xy)R_{2,3}(y) = R_{2,3}(y)R_{1,3}(xy)R_{1,2}(x) \tag{1}$$

as endomorphisms of $V \otimes V \otimes V$, where the two sub-indices of R indicate on which two tensor legs of $V \otimes V \otimes V$ it acts. For example, $R_{1,2}(x) = R(x) \otimes \mathrm{Id}_V$ and $R_{2,3}(x) = \mathrm{Id}_V \otimes R(x)$.

Example. Take V two dimensional with distinguished basis $\{u_+, u_-\}$. With respect to the ordered basis $\{u_+ \otimes u_+, u_+ \otimes u_-, u_- \otimes u_+, u_- \otimes u_-\}$ of $V \otimes V$,

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(1-x)v}{v^2-x} & \frac{v^2-1}{v^2-x} & 0 \\ 0 & \frac{(v^2-1)x}{v^2-x} & \frac{(1-x)v}{v^2-x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

is a solution of the quantum Yang-Baxter equation (1) for all $v \in \mathbb{C}$. This particular solution can be constructed using quantum groups (concretely, using the quantum affine algebra $\mathcal{U}_v(\widehat{\mathfrak{sl}_2})$, see e.g. [10, §9.6]), as well as using Hecke algebras, in which case v is the Hecke algebra parameter (we return to this at a later stage).

2000 *Mathematics Subject Classification.* Primary 32G34, 33D80; Secondary 33D52, 20C08.

The work of J. V. Stokman was supported by a VIDI-grant of the Netherlands Organization for Scientific Research (NWO).

Integrable systems. We discuss here two basic constructions that indicate the vital importance of solutions of quantum Yang-Baxter equations to integrable systems.

The first construction produces a generating function of *commuting* endomorphisms from a given invertible solution $R(x)$ of (1). For $N \in \mathbb{N}$, we consider the endomorphism

$$L(x) := R_{1,N+1}(x) \cdots R_{1,3}(x) R_{1,2}(x)$$

of $V \otimes V^{\otimes N}$. As before, the indices indicate the two tensor legs on which $R(x)$ acts in the $(N+1)$-fold tensor product $V \otimes V^{\otimes N}$. The transfer operator $\mathcal{T}(x)$ is the endomorphism of $V^{\otimes N}$ obtained by taking the trace of $L(x)$ over the first tensor leg of $V \otimes V^{\otimes N}$. The quantum Yang-Baxter equation satisfied by $R(x)$ implies the commutativity of the transfer operators,

$$\mathcal{T}(x)\mathcal{T}(y) = \mathcal{T}(y)\mathcal{T}(x).$$

Thus $\mathcal{T}(x)$ may be thought of as a kind of generating function (in x) of commuting endomorphisms of $V^{\otimes N}$. In physical applications, the commuting endomorphisms are interpreted as conserved quantities of some quantum spin chain. Applied for instance to the solution (2), the associated transfer operator $\mathcal{T}(x)$ produces conserved quantities for XXZ quantum spin chains; see e.g. [14].

The techniques and methods resulting from this construction, commonly referred to as the quantum inverse scattering method [16], have been successfully applied to numerous models, amongst which are various lattice models and quantum integrable field theories. It has led to the introduction [9] of fundamental new mathematical structures such as quantum groups, Yangians, and quantum affine algebras.

The second construction produces, from a given invertible solution $R(x)$ of (1), consistent systems of linear difference equations on the complex n-torus $T = \left(\mathbb{C}^\times\right)^n$. It is this construction that has played a fundamental role in the development of Cherednik's theory.

Fixing a quantum parameter $q \in \mathbb{C}^\times$, we write $\tau_j : T \to T$ for the q-dilation of the jth torus coordinate. With given endomorphisms $A_j(t)$ of a finite dimensional vector space U, depending meromorphically on $t \in T$, we now associate a system of linear q-difference equations on T by

$$f(\tau_j t) = A_j(t) f(t), \qquad j = 1, \ldots, n, \tag{3}$$

where f is a U-valued meromorphic function on T. The system (3) is said to be consistent if

$$A_i(\tau_j t) A_j(t) = A_j(\tau_i t) A_i(t), \qquad i, j \in \{1, \ldots, n\},$$

which, under mild conditions, is a sufficient condition for the existence of an ample supply of solutions of (3).

Given an invertible endomorphism L of V, we define endomorphisms

$$A_j(t) = R_{j,j-1}(qt_j/t_{j-1}) \cdots R_{j,1}(qt_j/t_1) L_j R_{n,j}(t_n/t_j)^{-1} \cdots R_{j+1,j}(t_{j+1}/t_j)^{-1} \tag{4}$$

of $U = V^{\otimes n}$, where L_j is the operator L acting on the jth tensor leg of $V^{\otimes n}$. The associated system (3) of linear q-difference equations is consistent if $R(x)$ satisfies the quantum Yang-Baxter equation (1) and if $L \otimes L$ commutes with $R(x)$. Under a mild additional condition on $R(x)$ (unitarity), a simple conceptual proof will present itself in a moment.

The above type of systems of q-difference equations arise naturally in physics as consistency equations for correlation functions of quantum spin chains (correlation functions encode essential physical properties of the models). For instance, using the particular solution (2), n-point correlation functions of the associated XXZ spin chain solve such a system of q-difference equations; see [14]. These types of q-difference equations first arose in work of Smirnov [25] on quantum integrable field theories, and in the work of Frenkel and Reshetikhin [11] as natural quantum analogs of the Knizhik-Zamolodchikov equations [15] (KZ equations for short). The latter equations form the system of linear differential equations satisfied by the n-point correlation functions of the Wess-Zumino-Novikov-Witten conformal field theory in two dimensions (see the monograph [10] for a detailed exposition).

The cocycle construction. The symmetric group S_n acts on $\mathbb{Z}^n$ via the permutation representation. The extended affine Weyl group is the semi-direct product group $W = S_n \ltimes \mathbb{Z}^n$. Several of the above mentioned key structures involving the quantum Yang-Baxter equation can be naturally formulated in terms of W-cocycles.

Let $\{\epsilon_j\}_{j=1}^n$ be the standard basis of $\mathbb{Z}^n$. The permutation action of S_n on T, together with $\epsilon_j \mapsto \tau_j$, defines a W-action on T. The W-cocycles we consider are maps $w \mapsto C_w(t)$ from W to a complex associative algebra H, depending meromorphically on $t \in T$, and satisfying

$$C_{ww'}(t) = C_w(t)C_{w'}(w^{-1}t), \qquad \forall\, w, w' \in W.$$

Denoting $C_i(t) = C_{s_i}(t)$ for the neighboring transposition $s_i \in S_n$ interchanging i and $i+1$, the cocycle condition entails the braid relations (with spectral parameter),

$$C_i(t)C_{i+1}(s_it)C_i(s_{i+1}s_it) = C_{i+1}(t)C_i(s_{i+1}t)C_{i+1}(s_is_{i+1}t) \tag{5}$$

for $i = 1, \ldots, n-1$. Furthermore, for a given finite dimensional complex H-module U, the associated endomorphisms $A_j(t) = C_{\epsilon_j}(\tau_j t)$ $(j = 1, \ldots, n)$ of U gives rise to a consistent system (3) of q-difference equations on T.

Suppose now that $R(x)$ is an invertible solution of the quantum Yang-Baxter equation (1) satisfying the unitarity condition $R(x^{-1}) = R_{2,1}(x)^{-1}$ (for example, (2) is unitary) and that L is an invertible endomorphism of V such that $R(x)$ commutes with $L \otimes L$. Let σ_i be the transposition of the ith and $(i+1)$st tensor leg of $V^{\otimes n}$. Then there exists a unique cocycle $w \mapsto C_w(t)$ with values in $\mathrm{End}(V^{\otimes n})$ such that

$$C_i(t) = \sigma_i R_{i,i+1}(t_{i+1}/t_i), \qquad C_\pi(t) = L_1\sigma_1\sigma_2\cdots\sigma_{n-1},$$

where $\pi := \tau_1 s_1 s_2 \cdots s_{n-1} \in W$ (we abuse notation by writing τ_j for ϵ_j, viewed as an element in W). In this situation, the braid relations (5) are a consequence of the quantum Yang-Baxter equation (1) for $R(x)$, and $A_j(t) = C_{\epsilon_j}(\tau_j t)$ is precisely (4).

Motivated by these observations, we now shift our focus to the construction of explicit W-cocycles, instead of solutions of the quantum Yang-Baxter equation.

Cherednik's cocycle. Cherednik's W-cocycle takes values in the extended affine Hecke algebra. The extended affine Hecke algebra $H = H_v$ is a complex, unital, associative algebra depending on $v \in \mathbb{C}^\times$. It is generated by $\{T_j\}_{j\in\mathbb{Z}/n\mathbb{Z}}$ and invertible

P with defining relations

$$\begin{aligned}
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \\
&T_iT_j = T_jT_i, \qquad\qquad i-j \neq \pm 1, \\
&(T_j - v)(T_j + v^{-1}) = 0, \\
&PT_{j-1} = T_jP.
\end{aligned}$$

Note that H is already algebraically generated by $T_1, \ldots, T_{n-1}$ and P. The subalgebra generated by T_j ($j \in \mathbb{Z}/n\mathbb{Z}$) is called the affine Hecke algebra.

For $v = 1$, the extended affine Hecke algebra H is the group algebra of W. In this case we denote the corresponding generators T_j by $s_j \in W$ ($j = 1, \ldots, n-1$) and P by $\pi \in W$; the affine Weyl group elements we already encountered in the previous paragraph.

The affine Hecke algebra is an indispensable tool in the representation theory of the general linear group $\mathrm{GL}(n;k)$ over a non-archimidean local field k. This is based on the fact [13] that the affine Hecke algebra, when v is the square root of the cardinality of the residue field of k, is isomorphic to the convolution algebra of complex, compactly supported $\mathcal{I}$-bi-invariant functions on $\mathrm{GL}(n;k)$, where $\mathcal{I}$ is the standard Iwahori subgroup of $\mathrm{GL}(n;k)$.

The algebra $\mathbb{C}[T]$ of regular functions on T consists of Laurent polynomials in the canonical coordinate functions z_i ($i = 1, \ldots, n$) on T. We view $\mathbb{C}[T]$ as a W-module by transposing the W-action on T. Cherednik [4] observed that the assignments

$$\begin{aligned}
&T_i \mapsto v + c_i(s_i - 1), \qquad i = 1, \ldots, n-1, \\
&P \mapsto \pi,
\end{aligned} \tag{6}$$

uniquely extend to an injective algebra embedding $\pi_{q,v} : H_v \to \mathrm{End}(\mathbb{C}[T])$, where c_i is the rational function

$$c_i = v^{-1}\left(\frac{1 - v^2 z_i z_{i+1}^{-1}}{1 - z_i z_{i+1}^{-1}}\right)$$

on T. The operators $\pi_{q,v}(T_i)$ are called Demazure-Lusztig operators.

Formally expressing s_i and π in terms of T_i and P in (6) leads to the elements

$$\begin{aligned}
&C_i(t) := c_i(t)^{-1}T_i + (1 - vc_i(t)^{-1}), \qquad i = 1, \ldots, n-1, \\
&C_\pi(t) := P,
\end{aligned} \tag{7}$$

in the extended affine Hecke algebra H_v, depending rationally on $t \in T$. Cherednik [2, 3] observed that (7) uniquely defines a (q-dependent) W-cocycle $\{C_w(t)\}_{w \in W}$ with values in the extended affine Hecke algebra H_v. In particular, for a given H_v-module U, the cocycle (7) produces solutions of the braid relations (5) in $\mathrm{End}(U)$. The system (3) of q-difference equations with $A_j(t) = C_{\epsilon_j}(\tau_j t)$ is consistent and is called the system of quantum affine Knizhnik-Zamolodchikov equations.

Example. We use the notations and conventions of the first example. The assignments $T_i \mapsto c_{i,i+1}$ ($i = 1, \ldots, n-1$) with

$$c = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & v - v^{-1} & 1 & 0 \\ 0 & 0 & 0 & v \end{pmatrix}$$

and $P \mapsto L_1\sigma_1\sigma_2\cdots\sigma_{n-1}$ with L an invertible endomorphism of V such that $L\otimes L$ commutes with c, define a H_v-representation on $V^{\otimes n}$. The resulting W-cocycle $C_w(t)$ is the one we constructed before using the particular solution $R(x)$ of the quantum Yang-Baxter equation from the first example.

The double affine Hecke algebra. A crucial ingredient in obtaining the W-cocycle (7) is Cherednik's double affine Hecke algebra $\mathcal{H}_{q,v}$ (of type GL_n). It is the subalgebra of $\mathrm{End}(\mathbb{C}[T])$ generated by $\mathbb{C}[T]$ (viewed as multiplication operators), by π, and by the Demazure-Lusztig operators $\pi_{q,v}(T_i)$; see (6). The cocycle can then be derived by inducing the regular representation of H_v to $\widetilde{\mathcal{H}}_{q,v}$, where $\widetilde{\mathcal{H}}_{q,v}$ is a suitable localization of the double affine Hecke algebra $\mathcal{H}_{q,v}$ (needed to be able to invert the rational functions c_i).

To understand the structure of the double affine Hecke algebra $\mathcal{H}_{q,v}$ better, we briefly recall some essential properties of the Bernstein-Zelevinsky presentation [18] for H_v. It characterizes the algebraic structure of H_v in terms of an abelian subalgebra $\mathcal{A}$ (isomorphic to $\mathbb{C}[T]$) and in terms of the subalgebra H_v^0 generated by $T_1,\ldots,T_{n-1}$. It is the Hecke algebra equivalent of the semi-direct product structure $W = S_n \ltimes \mathbb{Z}^n$ of the affine Weyl group. It plays a profound role in the representation theory of H_v.

The commutative algebra $\mathcal{A}$ is the algebra of Laurent polynomials in the (normalized) Y-operators $Y_i \in H_v$ $(i=1,\ldots,n)$. They can be obtained as limits of Cherednik's cocycle (7) by

$$Y_i = \lim_{t\to\infty} C_{\tau_i}(t),$$

where $t\to\infty$ means $|t_j/t_{j+1}|\to 0$ for $j=1,\ldots,n-1$. Granted the existence of the limit, the commutativity of the Y_i's is immediate from the cocycle condition!

A crucial property of the doube affine Hecke algebra is the linear isomorphism

$$\mathcal{H}_{q,v} \simeq \mathcal{A}\otimes H_v^0 \otimes \mathbb{C}[T],$$

called the Poincaré-Birkhoff-Witt property of $\mathcal{H}_{q,v}$, where the isomorphism is realized by multiplication of the tensor factors in $\mathcal{H}_{q,v}$. It is one of many properties of $\mathcal{H}_{q,v}$ suggesting that it is a natural generalization of the Weyl algebra. The fact that $\pi_{q,v}(H_v^0)$ and $\mathbb{C}[T]$ generate another copy of H_v in $\mathcal{H}_{v,q}$ justifies the terminology "double affine Hecke algebra".

Cherednik operators and Fourier transforms. The preceding paragraph leads to the following natural questions: Is there an automorphism of $\mathcal{H}_{q,v}$ that interchanges $\mathcal{A}$ and $\mathbb{C}[T]$ (and in fact, interchanges the two copies of H_v inside $\mathcal{H}_{q,v}$)? If so, is there a Fourier transform on $\mathbb{C}[T]$ that realizes this automorphism? Both questions are answered in the affirmative.

The key step is the diagonalization of the commuting subalgebra $\pi_{q,v}(\mathcal{A})$ of linear operators on $\mathbb{C}[T]$. The operators $\pi_{q,v}(Y_i)\subset \mathrm{End}(\mathbb{C}[T])$ $(i=1,\ldots,n)$ are pairwise commuting q-difference reflection operators on T, called Cherednik operators. The Cherednik operators are q-analogs of trigonometric Dunkl operators [1, 12]. The fact [1] that Dunkl operators are connected to (degenerate) affine Hecke algebras is one of the crucial insights which has led Cherednik to the introduction of the double affine Hecke algebra.

The Cherednik operators, acting on $\mathbb{C}[T]$, can be simultaneously diagonalized. The spectrum is simple for generic q and v. The common eigenfunctions P_λ, naturally parametrized by $\lambda\in\mathbb{Z}^n$, are called the nonsymmetric Macdonald polynomials

[20, 6]. Under suitable restrictions on q and v, they are orthogonal with respect to an explicit complex measure on a compact torus $T_u \subset T$, which is absolutely continuous with respect to the Haar measure on T_u. It allows one to define a Fourier transform $f \mapsto \widehat{f}$ for $f \in \mathbb{C}[T]$, where $\widehat{f}(\lambda)$ for $\lambda \in \mathbb{Z}^n$ is obtained by integrating fP_λ against the above mentioned measure. This Fourier transform realizes the automorphism interchanging the two copies of H_v in $\mathcal{H}_{q,v}$; in particular, it turns the Cherednik operators into multiplication operators on the space of finitely supported functions on $\mathbb{Z}^n$. The double affine Hecke algebra plays a key role in deriving the explicit Plancherel and inversion formulas of the Fourier transform (see [6]).

The Macdonald theory. By a symmetrization procedure, the nonsymmetric Macdonald polynomials turn into the famous S_n-invariant Macdonald polynomials P_λ^+. The Macdonald polynomials P_λ^+ [19, Chpt. VI] were introduced as interpolation between Schur functions, Jack polynomials, and Hall-Littlewood polynomials. They are common eigenfunctions for an explicit commuting family of q-difference operators called Ruijsenaars-Macdonald operators, which can in fact be obtained by symmetrizing the Cherednik operators. The Ruijsenaars-Macdonald operators were initially obtained by Ruijsenaars [24] as the conserved quantities of a relativistic version of the quantum integrable Calogero-Moser system, the latter describing pairwise interacting quantum particles on the circle.

On the other hand, solutions of the quantum affine KZ equation, considered as a system of q-difference equations with values in the principal series module of H_v, produce eigenfunctions of the Ruijsenaars-Macdonald operators by the so-called Cherednik-Matsuo correspondence [3]. This puts us right back to the first theme of the review!

In a famous preprint [21] from 1987 Macdonald generalized his polynomials to families of q-orthogonal polynomials associated to arbitrary root systems. The Macdonald polynomials discussed in this review are related to the root system of type A. The Macdonald polynomials associated to arbitrary root systems generalize the Weyl characters of simple Lie algebras as well as the elementary spherical functions on compact Lie groups and on simple p-adic Lie groups. They arise as elementary spherical functions on compact quantum symmetric spaces; see e.g. [23, 17].

The introduction [21] of the Macdonald polynomials associated to arbitrary root systems was complemented with a collection of deep conjectures, called the norm, evaluation, and duality conjectures. The norm conjecture generalizes the celebrated Macdonald constant term conjecture for root systems. A remarkable achievement of Cherednik is his proof of the Macdonald conjectures in a series [4, 5, 6] of papers using double affine Hecke algebras. Some essential steps of his proof of the conjectures have in fact been touched upon in this review. For instance, the norm conjecture turns out to be a direct consequence of the explicit inversion and Plancherel formula for the Fourier transform mentioned in the previous paragraph.

The book. The book consists of an introduction and three long chapters. Each chapter is essentially self-contained. In the remarkable introduction the author offers a personal and highly original view on various current developments in mathematics and physics which interact with the theory of double affine Hecke algebras. The discussion includes among others the following subjects: hypergeometric functions, Kazhdan-Lusztig theory, modular representation theory, fusion products, quantum groups, integrable lattice models, conformal field theory, and the geometric Langlands program. The author also pays attention to potentially important future developments in these directions.

The first chapter is based on lectures [7] of the author at Kyoto University in 1996–1997, and was written in collaboration with T. Akasaka, E. Date, K. Iohara, M. Jimbo, M. Kashiwara, T. Miwa, M. Noumi, Y. Saito and K. Takemura. This chapter discusses various generalizations of the KZ-equations, their connections to affine Hecke algebras and the correspondence to quantum many body problems (Cherednik-Matsuo correspondence). It amplifies the role of the (double) affine Hecke algebras in KZ-type connections, as we have discussed at some length for quantum KZ equations in the first part of this review.

Chapters 2 and 3 focus on the role of the double affine Hecke algebra in Fourier analysis and in Macdonald's theory (we scratched the surface of this in the second part of the review). Chapter 2 is based on a lecture series of the author at Harvard University in 2001, and the paper [8]. It discusses the rank one double affine Hecke algebra of type A in great length, together with its applications to Fourier analysis and Macdonald theory. In this case the Macdonald polynomials are the continuous q-ultraspherical polynomials, also known as Rogers polynomials, which date back to 1893. Chapter 3 discusses the generalization of Chapter 2 to general root systems. It concludes with a study of the representation theory of the double affine Hecke algebra.

It has become clear in recent years that Cherednik's double affine Hecke algebra and its degenerations have fundamental applications in a variety of directions. Two monographs on the subject have appeared: the book [22] by Macdonald and the book under review. These books nicely complement each other. Macdonald's book is entirely devoted to the applications of double affine Hecke algebras to the theory of Macdonald polynomials, leading up to complete proofs of the norm, evaluation, and duality conjectures. Written with great uniformity and clarity, it is a good account of the impressive algebraic and combinatorial power hidden behind Cherednik's definition of the double affine Hecke algebra. Cherednik's book on the other hand has a much wider scope, paying attention to the deep roots of the subject in mathematical physics as well as to the rich list of recent developments and applications of these new techniques. Both volumes are warmly recommended to all scholars and students who want to learn about this fascinating structure and its applications.

References

[1] I. Cherednik, *A unification of Knizhnik-Zamolodchikov equations and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411–432. MR1128220 (93b:17040)

[2] I. Cherednik, *Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators*, Internat. Math. Res. Notices **1992**, no. 9, 171–180. MR1185831 (94b:17040)

[3] I. Cherednik, *Induced representations of double affine Hecke algebras and applications*, Math. Res. Lett. **1** (1994), 319–337. MR1302647 (96i:17022)

[4] I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. of Math. (2) **141** (1995), no. 1, 191–216. MR1314036 (96m:33010)

[5] I. Cherednik, *Macdonald's evaluation conjectures and difference Fourier transform*, Invent. Math. **122** (1995), no. 1, 119–145. MR1354956 (98i:33027a)

[6] I. Cherednik, *Nonsymmetric Macdonald polynomials*, Internat. Math. Res. Notices **1995**, no. 10, 483–515. MR1358032 (97f:33032)

[7] I. Cherednik, *Lectures on Knizhnik-Zamolodchikov equations and Hecke algebras*. In: Quantum many-body problems and representation theory, 1–96, MSJ Mem., 1, Math. Soc. Japan, Tokyo, 1998. MR1724948 (2001g:20004)

[8] I. Cherednik, V. Ostrik, *From double affine Hecke algebra to Fourier transform*, Selecta Math. (N.S.) **9** (2003), no. 2, 161–249. MR1993484 (2004f:20011)

[9] V.G. Drinfel'd, *Quantum Groups*. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987. MR934283 (89f:17017)

[10] P.I. Etingof, I.B. Frenkel, A.A. Kirillov Jr., *Lectures on representation theory and Knizhnik-Zamolodchikov equations*. Mathematical Surveys and Monographs, 58. Amer. Math. Soc., Providence, RI, 1998. MR1629472 (2001b:32028)

[11] I.B. Frenkel, N. Yu. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60. MR1163666 (94c:17024)

[12] G.J. Heckman, *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. **103** (1991), 341–350. MR1085111 (92i:33012)

[13] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48. MR0185016 (32:2486)

[14] M. Jimbo, T. Miwa, *Algebraic analysis of solvable lattice models*, CBMS Regional Conference Series in Mathematics, 85. Amer. Math. Soc., Providence, RI, 1995. MR1308712 (96e:82037)

[15] V.G. Knizhnik, A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nuclear Phys. B **247** (1984), 83–103. MR853258 (87h:81129)

[16] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge Monographs on Mathematical Physics, Cambridge University Press 1993. MR1245942 (95b:81224)

[17] G. Letzter, *Quantum zonal spherical functions and Macdonald polynomials*, Adv. Math. **189**, no. 1 (2004), 88–147. MR2093481 (2005i:33019)

[18] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), no. 3, 599–635. MR991016 (90e:16049)

[19] I.G. Macdonald, *Symmetric functions and Hall polynomials*, second edition. Oxford Math. Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1005. MR1354144 (96h:05207)

[20] I.G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*. Séminaire Bourbaki, Vol. 1994/95. Astérisque No. **237** (1996), Exp. No. 797, 4, 189–207. MR1423624 (99f:33024)

[21] I.G. Macdonald, *Orthogonal polynomials associated with root systems*, Sém. Lothar. Combin. **45** (2000/01), Art. B45a, 40 pp. MR1817334 (2002a:33021)

[22] I.G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*. Cambridge Tracts in Mathematics, **157**. Cambridge University Press, Cambridge, 2003. MR1976581 (2005b:33021)

[23] M. Noumi, *Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math. **123**, no. 1 (1996), 16–77. MR1413836 (98a:33004)

[24] S.N.M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Comm. Math. Phys. **110** (1987), no. 2, 191–213. MR887995 (88i:58072)

[25] F. Smirnov, *Form factors in completely integrable models of Quantum Field Theory*, World Scientific, Singapore, 1992. MR1253319 (95a:81254)

Eric M. Opdam

KdV Institute for Mathematics, University of Amsterdam, The Netherlands
E-mail address: e.m.opdam@uva.nl

Jasper V. Stokman

KdV Institute for Mathematics, University of Amsterdam, The Netherlands
E-mail address: j.v.stokman@uva.nl