

Theory of finite simple groups, by Gerhard Michler, Cambridge University Press,
Cambridge, 2006, xii+662 pp., \$155.00, hardcover, ISBN 13: 978-0-521-86625-5

A group is called *simple* if it has more than one element and if its only normal subgroups are the trivial subgroup and the whole group. In other words, it is a group with precisely two normal subgroups. It is an elementary exercise to show that the abelian simple groups are precisely the finite cyclic groups of prime order. The nonabelian simple groups, on the other hand, are far more complicated.

There are many interesting examples of infinite simple groups, including examples with finite presentations, and the so-called “Tarski Monsters”, which have the highly counterintuitive property that their only proper nontrivial subgroups have order p for a fixed (large) prime number p . But there is little prospect of any kind of complete description of the infinite simple groups.

The finite simple groups are of more immediate interest to many mathematicians, firstly because they are more tractable, and secondly because, in their role as composition factors, they form the building blocks of all finite groups. The possibility that they might eventually be completely classified was first raised by Otto Hölder in 1892 and, around the turn of the 20th century, significant progress in the study of finite simple groups was made by William Burnside and Georg Frobenius in particular.

There followed a relatively inactive period, and work on this topic got underway again immediately after the Second World War, with the descriptions by Chevalley, Steinberg, Ree, and others of the simple groups of Lie type, and the classification theorems of Brauer, Suzuki, and Wall. But the contemporary study of finite simple groups really began in earnest in the early 1960’s, with the proof by Walter Feit and John Thompson that all finite nonabelian simple groups have even order [11], a result that had been conjectured much earlier by Burnside. Its exceptionally long and difficult proof set the scene for the following two decades of intense activity, culminating with the announcement in 1981 of the final outcome:

Theorem (The classification of finite simple groups). *Any finite simple group is isomorphic to one of the following:*

- (i) *A group of prime order;*
- (ii) *An alternating group of degree at least 5;*
- (iii) *A group of Lie type over a finite field;*
- (iv) *One of 26 sporadic groups.*

The groups in the first two of these categories are probably well known already to the reader. The groups of Lie type include the classical matrix groups over finite fields, that is, the linear, unitary, symplectic, and orthogonal groups. They also include the *exceptional groups of Lie type*, which occur as subgroups of certain classical groups and have a number of properties in common with the classical groups. As the name implies, the sporadic groups do not fit into any infinite families of groups. Five of these, the Mathieu groups M_{11} , M_{12} , M_{22} , M_{23} , and M_{24} were discovered by Emil Mathieu in the 19th century. The remainder have

2000 *Mathematics Subject Classification*. Primary 20D08; Secondary 20-04.

been discovered during the contemporary period, the first being the Janko group J_1 , found by Zvonimir Janko in 1966 [12].

The (alleged) proof of the classification theorem at the time of its announcement consisted of hundreds of thousands of pages of mathematics, much of it unpublished, by hundreds of mathematicians. Although it would be impossibly difficult to draw up a short list of, say, the ten leading players in this exercise, three names spring immediately to mind: John Thompson, for earlier work on the classification, Daniel Gorenstein (d. 1992), who was the principal and enduringly optimistic driving force behind the whole project, and Michael Aschbacher, who carried out so much of the hard work in the later stages, casually knocking off case after case, some of which had been given up by others as too difficult.

Of course, the proof as it was in 1981 was in a highly unsatisfactory state and the skepticism on the part of a significant proportion of the mathematical community might appear to have been justified by the discovery that the analysis of one case that had arisen (the so-called *quasithin* groups) was seriously deficient. This gap in the proof has only recently been securely filled by the papers [5] and [6] of Aschbacher and Smith, which has a total length of over 1200 journal pages.

Fortunately, a small but dedicated group of mathematicians began work in the early 1990's on rewriting the complete proof in a self-contained form that uses only standard and well-established results as a starting point, and avoids duplication. This project is expected to run eventually to 11 volumes amounting to a total of something in the region of 5000 pages. Six of these volumes have appeared to date, the latest being [9].

It is not the place here to discuss technical details of the classification itself. For an extensive history and survey, we refer the reader to Ron Solomon's article [19] and for a more recent and concise version to Aschbacher's article [4].

Of course, the study of finite simple groups did not end with their classification. While it is not sensible to expect any kind of complete description of all finite groups in terms of their simple composition factors (that would be hopeless even for groups of prime power order), various types of composite (i.e. not simple) groups that are closely related to the simple groups have been analyzed. In particular, the variants of the simple groups, including the *almost simple groups*, which are subgroups of the automorphism groups of finite nonabelian simple groups, and the *quasisimple groups*, which are groups G with $G = [G, G]$ and $G/Z(G)$ nonabelian simple play an essential role both in the classification itself and in the reduction of many problems involving finite groups in general.

Indeed, the classification has fundamentally changed the approach of many working mathematicians to problems in finite group theory. It has become more and more common to attack such problems by first attempting to reduce them to the case when the group is simple (or possibly almost simple, or quasisimple), and then to complete the analysis by examining the known simple groups on a case-by-case basis. This in turn has led to an intensified investigation and cataloguing of the properties of the simple groups and their variants, such as their representation theory and their subgroup and conjugacy structure. At an abstract level, the representation theory of groups of Lie type is most efficiently studied via the representation theory of algebraic groups. More concretely, the properties and character tables of the individual smaller simple groups can be found in printed form in the Atlas [1] and the Modular Atlas [3], and online at [2].

Finding the maximal subgroups of a finite group is a nice example of a problem that can be reduced in this way. General results proved in the 1980's enable the maximal subgroups of a finite group G with no nontrivial solvable normal subgroup to be described in a very precise and constructive form in terms of the maximal subgroups of the almost simple groups that occur as sections of G . Unfortunately there is no prospect of a complete description of all maximal subgroups of all almost simple groups. However, these maximal subgroups can be categorized and described at a higher level, and this in turn makes it possible to use existing lists of subgroups of groups and group representations to construct the required subgroups in a recursive fashion.

For example, sufficient data has been calculated that enables one to easily compute the maximal subgroups of the symmetric groups up to degree 2500. For the classical groups, the book by Kleidman and Liebeck [13] provides a very detailed description of the maximal subgroups of *geometric type*, which are, roughly speaking, those that preserve some structure, such as a subspace of the vector space on which the group acts. The remaining maximal subgroups are almost simple (modulo scalars) and can, in principle, be found using available lists of group representations that are complete up to dimension 250.

As the above discussion indicates, the theory of finite simple groups is an extensive and multifaceted area, and it abounds in very long, difficult, and highly technical proofs. One would naturally expect any book on this topic to deal principally only with some limited aspect of it, corresponding to the author's own interests. That is certainly the case with Gerhard Michler's book, *Theory of Finite Simple Groups*.

By a *uniqueness proof* of a finite simple group G , we shall mean a proof of a statement of the type "any finite simple group with certain specified properties is isomorphic to G ". Typically, these assumed properties concern the structure of certain subgroups of the group, such as the Sylow subgroups or the centralizer of an element of order two, but they could also include the order of the group or perhaps its character table.

The principal purpose of the book is to describe existence and uniqueness proofs of some of the sporadic simple groups, and the earlier chapters are devoted to developing those parts of the general theory that are required for this purpose. It seems exceedingly strange, however, that despite the fact that this aim involves one important aspect of the classification, the classification of finite simple groups itself is not mentioned anywhere in the text.

The proofs described in the book involve extensive computer calculations. Group theory has always been a computational subject, and the earliest algorithms proposed for solving various group-theoretical problems, such as finding the order of a finite group defined by a finite presentation, predate the days of mechanical computers. Solving problems in group theory was also one of the earliest nonnumerical applications of programmable computers, which date back to the early 1950's. The first really major application of computers to finite simple groups was the proof by Charles Sims in 1972 of the existence of the Lyons sporadic simple Ly group, which he constructed as a permutation group on 8 835 156 points [18]. Nowadays, the Computer Algebra Systems GAP and MAGMA provide mathematicians lacking expertise in the use of computers with convenient access to a wide range of high-quality implementations of the fundamental algorithms in group theory, together

with programming languages suitable for implementing new algorithms. The reviewer's book [10] is devoted to computational group theory, and [17] contains a more specialized treatment of computing in a finite permutation group.

It should be stressed, however, that the proof of the classification of finite simple groups is almost completely computer free, and the remaining computer-dependent parts are probably restricted to the existence and uniqueness of a small number of the sporadic groups. The largest of the sporadic groups, the "Monster", was constructed without computer assistance by Griess in 1982 [8].

Computer calculations have, however, played a very central role in the study and calculation of the properties of the simple groups themselves, particularly those of small to moderately large order. Most of the data to be found in the Atlas, the Modular Atlas, and the online Atlas was computed mechanically, and the databases in which this data is stored are accessible via GAP and MAGMA. Character tables can already be computed quickly for groups of large order, and considerable effort is being devoted to compiling databases of characteristic zero representations of the simple groups and their variations. This data is used, in particular, by algorithms for computing the maximal subgroups of a general finite group.

Turning now to the book under review, the first four chapters are in the style of a standard graduate level textbook, and they provide a formal treatment of some group theory and ordinary and modular representation theory that is required for the later chapters and is generally relevant to the analysis of finite simple groups. Although the classification theorem itself is not mentioned, four of the major individual results that played an essential role in the early stages of the classification, including the Feit-Thompson Odd Order Theorem that we have already mentioned, are cited.

The following result, which is proved in Chapter 2, plays a central role in the applications to be described later. An *involution* is a group element of order two.

Theorem (Brauer-Fowler). *Let G be a finite simple group having an involution z such that $|C_G(z)| = n$. Then G can be embedded into the alternating group A_{n^2-1} of degree $n^2 - 1$. In particular, there are only finitely many nonisomorphic simple groups G having an involution z such that $|C_G(z)| = n$.*

The later chapters in the book are more in the style of a monograph with specific aims in mind. Chapters 5 and 6 introduce some basic algorithmic techniques from computational group theory. There are a number of nonstandard attributions to be found in this book. For example, one of the first technical tricks used by Sims in the 1970's to construct large sporadic groups as finite permutation groups is described as the "Gollan double coset trick".

Chapter 7 is devoted to the description of an algorithm, due the book's author, which is essentially a constructive version of the Brauer-Fowler Theorem. Its aim is to compute (up to isomorphism) all finite simple groups G with an involution z , central in a Sylow 2-subgroup of G , such that the centralizer $H = C_G(z)$ of z has a given structure. There are a few other technical restrictions on G , which we shall not list in detail here. The algorithm is based on a theorem of Thompson about amalgamated free products, and similar methods had been used previously by Norton to construct Janko's sporadic group J_4 , and by Parker and Wilson [16].

The remaining chapters describe successful applications of this algorithm. Chapter 8 contains a verification of all known examples of groups H for which there is

more than one isomorphism class of finite simple groups having H as an involution centralizer. Chapters 9, 10, 11, and 12 concern the use of the algorithm to prove existence and uniqueness of the first Janko group J_1 , the Higman-Sims group HS, the Harada-Norton group $Ha = HN (= F_5)$, and the Thompson group Th ($= F_3$), respectively. These individual results have also been published by Michler and collaborators (Kratzer, Previtoli and Weller) in separate research papers, as have similar proofs for other sporadic groups, including the O’Nan group ON and the McLaughlin group McL.

A negative aspect (for some) of this method is that it involves very large-scale computer calculations, for which the majority of interested readers would have neither the technical ability nor the available hardware resources to reproduce. I suspect that this remains true despite the fact that, for the largest of these calculations on the group Th, a DVD containing technical data to facilitate the process of checking the calculations has been supplied with the book. This calculation involves computing the order of a permutation representation of Th of degree 143 127 000. Another possible criticism is that the calculations for the smaller examples, such as HS, result in proofs that are considerably more complicated than older existing proofs both with and without computer. I am sure that all interested parties would agree that it would, at least in principle, be desirable to have computer-free proofs of all results of this type.

On the other hand, the sporadic simple groups have lived up to their name by resolutely resisting all attempts to provide uniform descriptions of them. The approach described and advocated by Michler in this book is remarkable in that it offers a uniform method that can, in principle, be used to prove computationally the existence and uniqueness of almost all of the sporadic groups. Indeed, according to one of his earlier papers [15], the smallest Mathieu group M_{11} is the only sporadic group for which this method could not work. Furthermore, if, as seems likely at present, a few of the results of this type are going to remain dependent on computer calculations, then it is highly desirable for them to be checked independently using a variety of different approaches.

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