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The arithmetic of dynamical systems, by Joseph H. Silverman, Graduate Texts in Mathematics, 241, Springer, New York, 2007, x+511 pp., US$49.95, hardcover, ISBN 13: 0-978-387-69903-5

The field of dynamical systems traces its roots to Poincaré's qualitative study of solutions to differential equations in the late nineteenth century. The subfield of complex dynamics, which was initiated by Fatou and Julia in the late 1910s but which did not draw substantial attention until the 1980s, focuses mainly on the iteration of rational functions. In the 1990s, number theorists began to study such iterations as well, noting parallels to certain aspects of the theory of elliptic curves. Since then, number-theoretic dynamics has begun to emerge as a field in its own right, especially concerning the rationality properties of periodic points.

1. Rational preperiodic points

A (discrete) dynamical system is a set X and a function $\phi : X \to X$; one considers the iterates $\phi^n := \phi \circ \phi \circ \cdots \circ \phi$ for integers $n \geq 0$. We say a point $x \in X$ is *periodic* if $\phi^n(x) = x$ for some $n \geq 1$; if n is the smallest such integer, the set $\{x, \phi(x), \ldots, \phi^{n-1}(x)\}$ is called an *n-cycle*. More generally, given $x \in X$, if there is an integer $m \geq 0$ such that $\phi^m(x)$ is periodic, we say x is *preperiodic*.

Complex dynamics is devoted mostly to the case that X is the complex projective line (i.e., Riemann sphere) $X = \mathbb{P}^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$, and that ϕ is a rational function with coefficients in $\mathbb{C}$. Motivated by number theory's focus on rational numbers, one can instead ask about (rational) preperiodic points in $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$, under the action of a rational function $\phi \in \mathbb{Q}(z)$. That is, we wish to study the set

$$\mathrm{Preper}(\phi, \mathbb{Q}) := \{x \in \mathbb{P}^1(\mathbb{Q}) : x \text{ is preperiodic}\}.$$

Example 1.1. Let us compute $\mathrm{Preper}(\phi, \mathbb{Q})$ for $\phi(z) = z^2 - 1 \in \mathbb{Q}[z]$. First, note that ∞ itself is fixed by ϕ; indeed, for any polynomial, ∞ is always a fixed point, and no other points map to it. Second, consider $x = m_0/n_0 \in \mathbb{Q}$, where m_0 and n_0 are relatively prime integers and $n_0 \geq 1$. If $n_0 \geq 2$, then there is some prime p and some integer $r \geq 1$ such that $p^r || n_0$; that is, p^r divides n_0 but p^{r+1} does not. A simple calculation shows that when we write $\phi(x) = m_1/n_1$ in lowest terms, we must have $p^{2r} || n_1$. Proceeding inductively, the denominator n_k of $\phi^k(x)$ satisfies $p^{2^k r} || n_k$. The iterates of x are therefore all distinct. Thus, if $x \in \mathbb{Q}$ is to be preperiodic, its denominator n_0 must be 1; that is, x must in fact be an integer.

On the other hand, suppose $x \in \mathbb{Q}$ satisfies $|x| \geq 2$. Then $\phi(x) \geq 3$, and in fact $\phi(x) > |x|$. The iterates of x then form a strictly increasing sequence, and once again, x cannot be preperiodic. Thus, the only points of $\mathbb{P}^1(\mathbb{Q})$ that could possibly be preperiodic are ∞, -1, 0, and 1. A simple computation shows

$$1 \longmapsto 0 \longmapsto -1 \longmapsto 0,$$

so that 0 and -1 form a 2-cycle, 1 is preperiodic but not periodic, and, of course, ∞ is fixed.

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Example 1.2. Let us now find $\mathrm{Preper}(\phi, \mathbb{Q})$ for $\phi(z) = z^2 - 133/144$. (This seemingly obscure choice comes from setting $\mu = 5$ in Theorem 2 of [12].)

Given $x = m_0/n_0 \in \mathbb{Q}$ in lowest terms, suppose $p^r || n_0$ for some prime $p \geq 5$ and integer $r \geq 1$; then the argument of Example 1.1 shows that x is not preperiodic. The same conclusion follows if $2^r || n_0$ for some $r \neq 2$ or if $3^s || n_0$ for some $s \neq 1$. Therefore, if $x \in \mathbb{Q}$ is preperiodic, its denominator must be exactly 12.

Once again, meanwhile, a preperiodic point $x \in \mathbb{Q}$ must satisfy $|x| < 2$, or else the iterates of x will explode in size. Thus, x must be one of the sixteen rational numbers in the interval $(-2, 2)$ with denominator exactly 12. This time, however, not all of those numbers are actually preperiodic. For example,

$$\frac{17}{12} \longmapsto \frac{13}{12} \longmapsto \frac{1}{4},$$

and because $1/4$ has the wrong denominator, neither $13/12$ nor $17/12$ is preperiodic. Similarly, $5/12$ and $23/12$ are not preperiodic. Still, it is easy to check that $\pm 1/12$, $\pm 11/12$, $\pm 7/12$, and $\pm 19/12$ are preperiodic. Specifically, $-1/12$ and $-11/12$ form a 2-cycle; $-7/12$ and $19/12$ are fixed; and clearly $\phi(-x) = \phi(x)$ for all x. Together with the fixed point at ∞, then, ϕ has nine preperiodic points in $\mathbb{P}^1(\mathbb{Q})$.

The two examples above suggest the following result, proven by Northcott in 1950 [11]. To state it, we define the *degree* of a rational function $\phi \in \mathbb{Q}(z)$ to be the maximum of the degrees of the numerator and denominator.

Theorem 1.3. *Let $\phi \in \mathbb{Q}(z)$ be a rational function of degree $d \geq 2$. Then ϕ has only finitely many rational preperiodic points. That is, $\#\,\mathrm{Preper}(\phi, \mathbb{Q}) < \infty$.*

One of the major conjectures of arithmetic dynamics, proposed by Morton and Silverman in 1994 [10], is the following Uniform Boundedness Conjecture.

Conjecture 1.4. *Let $d \geq 2$. Then there is a constant $C = C_d$ such that for any rational function $\phi \in \mathbb{Q}(z)$ of degree d,*

$$\#\,\mathrm{Preper}(\phi, \mathbb{Q}) \leq C_d.$$

In fact, Northcott proved his result, and Morton and Silverman stated their conjecture, not just for the projective line over $\mathbb{Q}$, but for projective N-space (for any $N \geq 1$) and over any *number field*, i.e., any finite extension of $\mathbb{Q}$. For the sake of exposition, this review will stick (mostly) with $\mathbb{P}^1(\mathbb{Q})$, but the reader should note that everything we say can be done over number fields, and quite a lot of it can be done in higher dimension.

Part of the evidence for Conjecture 1.4 comes from an analogy with the uniform boundedness of torsion on elliptic curves over number fields (proven by Mazur [6] for $\mathbb{Q}$, and ultimately proven in full generality by Merel [7]). There is also fairly strong evidence that Conjecture 1.4 is true at least for quadratic polynomials over $\mathbb{Q}$, with an upper bound of $C = 9$, as attained in Example 1.2 above; see [12]. Still, a full resolution of the conjecture, even in the simplest case of quadratic polynomials over $\mathbb{Q}$, seems to be out of reach at the moment.

2. Heights and canonical heights

At its core, the proof of Northcott's theorem is a generalization of the arguments of Examples 1.1 and 1.2, using Weil's theory of Diophantine heights, which begins with the following simple idea.

Any $x \in \mathbb{Q}$ can of course be written as a fraction m/n in lowest terms (with $m, n \in \mathbb{Z}$), in which case we define the *height* of x to be simply

$$h(x) = h(m/n) := \log \max\{|m|, |n|\}. \tag{2.1}$$

(In order to define h on all of $\mathbb{P}^1(\mathbb{Q})$, we may treat ∞ as $1/0$, so that $h(\infty) = 0$.) Intuitively, $h(x)$ is the amount of space x would take up on the page if written out as a fraction; put another way, it measures some sort of arithmetic complexity of the rational number x. Clearly, $h(x)$ is independent of the choice of m and n (which can only vary up to sign). Moreover, for any given bound $B \in \mathbb{R}$, the set

$$\{x \in \mathbb{P}^1(\mathbb{Q}) : h(x) \leq B\}$$

is finite.

The connection with dynamics comes from the following much less obvious property of the height function.

Theorem 2.1. *Let $\phi \in \mathbb{Q}(z)$ be a rational function of degree $d \geq 1$. Then there is a constant $C_\phi \in \mathbb{R}$ such that for* any $x \in \mathbb{P}^1(\mathbb{Q})$,

$$|h(\phi(x)) - d \cdot h(x)| \leq C_\phi.$$

The conclusion of this fundamental result from Diophantine geometry is often stated in "big-Oh" notation as the approximate functional equation

$$h(\phi(x)) = d \cdot h(x) + O_\phi(1). \tag{2.2}$$

The $\leq$ direction of (2.2) is actually fairly easy to prove, but the $\geq$ direction is far more difficult. Given Theorem 2.1, however, it is easy to deduce that if $d \geq 2$, then no $\mathbb{Q}$-rational preperiodic point can have height more than $C_\phi/(d-1)$. Northcott's theorem for $\mathbb{P}^1(\mathbb{Q})$ then follows from the finiteness of sets of bounded height.

Note that if $\phi(z) = z^d$, then the approximate formula $h(\phi(x)) = d \cdot h(x) + O(1)$ is in fact the exact equality $h(\phi(x)) = d \cdot h(x)$. More generally, if $\deg \phi = d \geq 2$, we can define the *canonical height* for ϕ by

$$\hat{h}_\phi(x) = \lim_{n\to\infty} d^{-n} h(\phi^n(x)).$$

It is easy to check that this limit converges, and that the *exact* functional equation $\hat{h}_\phi(\phi(x)) = d \cdot \hat{h}_\phi(x)$ holds. In addition, $\hat{h}_\phi(x) = h(x) + O_\phi(1)$, so that as before, there are only finitely many rational points of canonical height below any given bound. At the same time, $\hat{h}_\phi(x)$ (which is always nonnegative) is zero if and only if x is a preperiodic point of ϕ. Thus, the canonical height function $\hat{h}_\phi$ combines both arithmetic information (it approximates h, the arithmetic complexity) and dynamical information (returning zero precisely at preperiodic points).

3. Places and reduction

As is common in number theory, one can gain more insight by the use of absolute values. An *absolute value* on a field K is a function $|\cdot| : K \to [0, \infty)$ that is nondegenerate ($|x| = 0$ if and only if $x = 0$), multiplicative ($|xy| = |x||y|$), and satisfies the triangle inequality ($|x+y| \leq |x| + |y|$). Besides the usual absolute value, which we will denote $|\cdot|_\infty$ hereafter, there is a p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ for each prime number p. The latter is defined by $|p^r q|_p = p^{-r}$, where $q \in \mathbb{Q}$ is a rational number with neither numerator nor denominator divisible by p, and $r \in \mathbb{Z}$. It is a *non-archimedean* absolute value, meaning that it satisfies the strengthened triangle inequality $|x+y|_p \leq \max\{|x|_p, |y|_p\}$.

In the early twentieth century, Ostrowski proved that the standard absolute value and the p-adic absolute values are essentially the only absolute values on $\mathbb{Q}$. The set $M_{\mathbb{Q}} := \{|\cdot|_\infty, |\cdot|_2, |\cdot|_3, |\cdot|_5, \dots\}$ satisfies the *product formula*

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1 \qquad \text{for all nonzero } x \in \mathbb{Q},$$

where we have abused notation by sometimes writing v in place of $|\cdot|_v$. Number theorists often think of $M_{\mathbb{Q}}$ as a geometric object whose elements are called the *places* of $\mathbb{Q}$. The idea is that one can work *locally* by considering a particular absolute value $|\cdot|_v$ and then try to combine the local information to prove *global* results—that is, results about $\mathbb{Q}$ itself.

Returning to heights, it is an easy consequence of the product formula that

$$h(x) = \sum_{v \in M_{\mathbb{Q}}} \log \max\{1, |x|_v\} \quad \text{for all } x \in \mathbb{Q}. \tag{3.1}$$

Note that for any $x \in \mathbb{Q}$, all but finitely many of the terms on the right side are zero, and hence this seemingly infinite sum is in fact finite. Note also that this formula no longer requires us to write x as a fraction in lowest terms; indeed, to define the height for arbitrary number fields, where the ring of integers might not have unique factorization, one usually uses a version of (3.1) rather than (2.1).

The term $\lambda_v(x) = \log \max\{1, |x|_v\}$ in (3.1) is often called the *local height* of x (at v). If we write $\phi(x) = f(x)/g(x)$, where f and g are polynomials, then λ_v, like h, also satisfies an approximate functional equation, this time of the form

$$\lambda_v(\phi(x)) = d \cdot \lambda_v(x) - \log|g(x)|_v + O_{\phi,v}(1),$$

where the "big-Oh" constant depends on ϕ and v but not on x. Moreover, for any given ϕ, the "big-Oh" constant is zero for all but finitely many v. That is, for such v,

$$\lambda_v(\phi(x)) = d \cdot \lambda_v(x) - \log|g(x)|_v \tag{3.2}$$

for all x such that $g(x) \neq 0$. This last fact is intimately related to the fundamentally important notion of *good reduction*, as follows.

Given $\phi(z) = f(z)/g(z) \in \mathbb{Q}(z)$ and a prime number p, we would like to be able to reduce ϕ modulo p. That is, simply replace all the coefficients of f and g by their images in the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of p elements, to get a rational function

$$\overline{\phi}(z) = \frac{\overline{f}(z)}{\overline{g}(z)} \in \mathbb{F}_p(z).$$

To do that, of course, we must first clear denominators in f and g so that all coefficients lie in $\mathbb{Z}$; it would also make sense to cancel any common prime factors among all those coefficients while we are at it. Thus, we may assume that f and g are relatively prime polynomials in $\mathbb{Z}[z]$. Even after making that assumption, however, there is the possibility of introducing some extra cancellations when we move from f and g to $\overline{f}$ and $\overline{g}$. Thus, we state the following definition.

Definition 3.1. Let $\phi(z) \in \mathbb{Q}(z)$ be a rational function of degree $d \geq 0$, and write $\phi(z) = f(z)/g(z)$, where $f, g \in \mathbb{Z}[z]$ are relatively prime polynomials. Let p be a prime number. Let $\overline{\phi}(z) = \overline{f}(z)/\overline{g}(z)$, where $\overline{f}, \overline{g} \in \mathbb{F}_p[z]$ are the reductions of f and g modulo p. If $\deg(\overline{\phi}) = \deg \phi$, then we say ϕ has *good reduction* at p, and that p is a *good prime* for ϕ. Otherwise, $\deg(\overline{\phi}) < \deg(\phi)$, and we say ϕ has *bad reduction* at p, and that p is a *bad prime* for ϕ.

For the sake of completeness, we also say that every $\phi \in \mathbb{Q}(z)$ has bad reduction at the archimedean place $v = \infty$.

If $\phi(z) \in \mathbb{Q}[z]$ is a polynomial, it is not difficult to show that ϕ has good reduction at a prime p if and only if p divides neither the numerator of the lead coefficient nor the denominators of any of the coefficients. For example, the map $\phi(z) = z^2 - 133/144$ may be written as f/g for $f(z) = 144z^2 - 133$ and $g(z) = 144$; note that $f, g \in \mathbb{Z}[z]$ are relatively prime. Thus, ϕ has bad reduction at 2 and 3, because modulo either prime, the degree of $\overline{\phi} = \overline{f}/\overline{g} = -1/0 = \infty$ has dropped from 2 to 0. By fiat, ϕ also has bad reduction at $v = \infty$. On the other hand, ϕ has good reduction at every other prime (even those dividing 133). Naturally, it is not a coincidence that the bad places 2, 3, and ∞ are precisely the ones that required special consideration in Example 1.2.

As an example of a nonpolynomial, consider $\psi(z) = (3z^2 - 8z)/(4z + 11) \in \mathbb{Q}(z)$. One can check that ψ has bad reduction at $v = 3, 5, 11, 13, \infty$, and good reduction everywhere else. For example, modulo 5, we have $\overline{\psi}(z) = 3z(z-1)/(-z+1) = -3z$; the degree has dropped because of a cancellation introduced by reduction modulo 5.

In general, the reduced map $\overline{\phi}$ of course acts on $\mathbb{P}^1(\mathbb{F}_p) = \mathbb{F}_p \cup \infty$. Meanwhile, there is a natural reduction map from $\mathbb{P}^1(\mathbb{Q})$ to $\mathbb{P}^1(\mathbb{F}_p)$, sending x to $x(\operatorname{mod} p)$ if $|x|_p \leq 1$, and sending every x with $|x|_p > 1$ to ∞. This reduction map, which we will denote by $x \mapsto \overline{x}$, can also be extended to a map from $\mathbb{P}^1(\overline{\mathbb{Q}})$ to $\mathbb{P}^1(\overline{\mathbb{F}}_p)$, where $\overline{K}$ denotes the algebraic closure of the field K.

Theorem 3.2. *Let $\phi(z) \in \mathbb{Q}(z)$ be a rational function of degree $d \geq 0$, and write $\phi(z) = f(z)/g(z)$, where $f, g \in \mathbb{Z}[z]$ are relatively prime polynomials. Then*

(1) *ϕ has only finitely many bad primes.*
(2) *Let p be a prime number. The following are equivalent:*
   (a) *ϕ has good reduction at p.*
   (b) *$\overline{\phi}(\overline{x}) = \overline{\phi(x)}$ for all $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$, where $x \mapsto \overline{x}$ denotes the reduction modulo p map from $\mathbb{P}^1(\overline{\mathbb{Q}})$ to $\mathbb{P}^1(\overline{\mathbb{F}}_p)$.*
   (c) *$\lambda_p(\phi(x)) = d \cdot \lambda_p(x) - \log|g(x)|_p$ for all $x \in \overline{\mathbb{Q}}$ such that $g(x) \neq 0$.*

Part (1) of Theorem 3.2 holds because all the bad primes divide the resultant of f and g, which is an integer. More precisely, the resultant is a certain polynomial (the Sylvester determinant) in the coefficients of f and g. Meanwhile, condition (b) of part (2) above says simply that it makes sense to work with ϕ modulo p; and condition (c) is exactly equation (3.2).

Of course, at the finitely many places v at which ϕ has bad reduction, the analysis is far more complicated. To understand the dynamics at a bad place v, we should consider the completion of $\mathbb{Q}$ at v; that is, toss in all limits of sequences that are Cauchy with respect to $|\cdot|_v$, to obtain $\mathbb{Q}_\infty := \mathbb{R}$ if $v = \infty$, or $\mathbb{Q}_p$, the field of p-adic rationals, if $v = p$ is a prime.

It is often useful to work over the algebraic closure, too. Thus, for $v = \infty$, we turn to $\mathbb{C}$, and we are back in the setting of complex dynamics, studying the iteration of a rational function on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Complex dynamics, while still a very fertile field, has fortunately been around long enough that there are numerous expositions on the subject, such as [1, 4, 8]. In particular, the Fatou and Julia sets of ϕ in $\mathbb{P}^1(\mathbb{C})$, as well as the classification of periodic points as attracting, repelling, or indifferent, are understood well enough to shed light on problems over $\mathbb{Q}$. The bounds $|x|_\infty < 2$ for preperiodic points in Examples 1.1 and 1.2, for instance, come from an understanding of the Fatou and Julia sets of the maps in question.

On the other hand, if $v = p$ is prime, then the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is not complete, but its completion $\mathbb{C}_p$ is both complete and algebraically closed. We can therefore consider the action of ϕ on $\mathbb{P}^1(\mathbb{C}_p)$, the p-adic analogue of the Riemann sphere. Dynamics over non-archimedean fields is a newer research area with fewer expositions, although some early research papers such as [2, 3, 5, 13] provide something of an introduction to the subject. Nevertheless, much of the same theory of Fatou and Julia sets can be carried out in the non-archimedean context; the denominators of 12 in Example 1.2, for instance, can be understood as coming from the 2-adic and 3-adic Julia sets of $\phi(z) = z^2 - 133/144$.

Given these fundamentals (canonical heights, reduction, complex dynamics, and non-archimedean dynamics), one can of course study many arithmetic aspects of dynamics other than the rationality of preperiodic points. For example, as is conjectured for elliptic curves, is there a positive lower bound, depending only on some kind of conductor of ϕ, for the canonical height of a nonpreperiodic point? If we fix ϕ but let our number field K grow, how does the set of K-rational preperiodic points grow? If ϕ is *not* a polynomial, is there an upper bound on the number of consecutive iterates any $x \in \mathbb{P}^1(\mathbb{Q})$ can have that are all *integers*, assuming some appropriate minimality restrictions on our choice of coordinates? What do various moduli spaces of dynamical systems look like? These and many other questions, including questions about canonical heights, non-archimedean dynamics, and other specialized tools, remain very much open.

4. This book

Silverman presents a thorough introduction to the dynamics of a rational function $\phi(z) \in K(z)$ acting on $\mathbb{P}^1(K) := K \cup \{\infty\}$, where K is a number field. *The Arithmetic of Dynamical Systems* develops the relevant aspects of heights, canonical heights, complex dynamics, and non-archimedean dynamics from scratch. While the reader is assumed to be familiar with number fields, Galois theory, some algebraic number theory, and a smattering of algebraic geometry, the exposition is essentially self-contained. The book is aimed at an audience of number theorists and their graduate students; in particular, no previous knowledge of dynamical systems is assumed. In addition, although the reader is of course referred to the (extensive) bibliography for the proofs of some of the deeper results, most of the many lemmas and theorems are proven in full detail in the text.

For example, Silverman devotes one chapter to complex dynamics and two (one for good reduction, one for bad) to non-archimedean dynamics. Topics in each case include multipliers of periodic points, Fatou and Julia sets, and coordinate changes (i.e., conjugation by linear fractional transformations). The discussion of non-archimedean dynamics also includes a brief and very readable introduction to Berkovich space and dynamics on the Berkovich projective line; readers wishing to learn that theory may find it helpful to start with Silverman's book before turning to the more comprehensive discussions in [13] or [14], for example. Another chapter, devoted to dynamics over $\mathbb{Q}$ and other number fields, develops height and canonical height functions from the ground up. The same chapter also discusses integrality (cf. the aforementioned question on consecutive integer iterates) and Diophantine approximation à la Siegel and Roth, as well as the action of Galois on preperiodic points.

Other chapters branch out towards a small selection of more advanced topics. One such topic is the study of moduli spaces of dynamical systems. Here, Silverman defines *dynatomic polynomials*, analogous to cyclotomic polynomials, as certain polynomials whose roots are periodic points of a given map ϕ. He uses them to define certain moduli spaces that can be considered analogous to the classical modular curves from the theory of elliptic curves. He also studies the full moduli space $\mathcal{M}_d$ of all degree-d rational maps; $\mathcal{M}_d$ is in general a highly nontrivial object because it parametrizes not individual maps ϕ but rather equivalences classes of maps ϕ under coordinate change.

Another chapter is devoted mainly to Lattès maps, which arise from endomorphisms of elliptic curves. Silverman devotes one section to a high-speed review of the relevant aspects of the theory of elliptic curves, but his well-known text [15] would still be advisable as background for the reader interested in this topic. Finally, the last chapter presents an introduction to number-theoretic dynamics in higher dimension, especially the case of automorphisms either of affine space $\mathbb{A}^n$ or of a K3 surface.

As Silverman notes in the introduction, there are of course many other topics that can be considered arithmetic or algebraic dynamics. The $3n+1$ problem is probably the most famous such topic; others include dynamics over other algebraic fields or on Drinfeld modules. Even within the study of dynamics of algebraic morphisms over local and global fields, there are many active areas of research that Silverman's book mentions but does not really cover, such as ergodic theory in p-adic dynamics (requiring the Berkovich theory) and Galois equidistribution of preperiodic points.

The Arithmetic of Dynamical Systems arrives with auspicious timing. The subject has now been studied long enough and by enough researchers that a fairly broad literature exists, the field has a "big picture" to guide it, and numerous open questions have been posed to motivate future research. At the same time, the field is young enough that there are few, if any, other such comprehensive introductions to the subject. With a growing number of graduate students and established researchers trying to learn the subject, such a clear exposition comes none too soon.

The book is well organized and well written, with a style and structure similar to those of Silverman's elliptic curves texts [15, 16]. Ideas and intuitions are conveyed clearly, but at the same time, the presentation is completely rigorous. There are hundreds of exercises at a wide range of difficulty levels, and there are dozens of conjectures and open questions embedded in the exposition. Number theorists interested in studying dynamics will find this book to be both an excellent introduction and a valuable reference for the subject.

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