BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 1, January 2009, Pages 133–135
S 0273-0979(08)01239-1
Article electronically published on October 10, 2008

SELECTED MATHEMATICAL REVIEWS

In memory of Oded Schramm

MR1776084 (2001m:60227) 60K35; 30C85, 60H15, 82B27, 82B44

Schramm, Oded

Scaling limits of loop-erased random walks and uniform spanning trees.

Israel J. Math. **118** (2000), 221–288.

This paper makes two contributions to the understanding of continuum scaling limits of lattice random systems. The first is a complete characterization of the topology of the loop-erased random walk (LERW) and the closely related uniform spanning tree (UST) in the lattice $\delta\mathbf{Z}^2$ as $\delta \to 0$. It is shown that both processes have scaling limits at least along subsequences $\delta = \delta_n \to 0$. Any scaling limit of LERW is supported on simple curves. Any scaling limit of UST is supported on topological trees which are dense in the plane and which consist of simple curves that meet at branch points of degree three. The proofs combine D. B. Wilson's algorithm [in *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*, 296–303, ACM, New York, 1996; see MR1427491 (97g:68005)] with geometric estimates; the analysis of UST also makes essential use of self-duality.

The second contribution is the discovery of a new family of conformally invariant stochastic processes, the stochastic Loewner evolution processes (SLE). This is a far-reaching result, because many critical models in the plane are expected to have conformally invariant scaling limits. The SLE processes are natural candidates for the laws of such scaling limits. The values of the SLE processes are conformal deformations of the unit disc into a subset which shrinks with time. Such deformations are described by C. Loewner's differential equation [Math. Ann. **89** (1923), 103–121; JFM 49.0714.01]; here, the driving process is a suitably rescaled Brownian motion on the boundary of the disc. The analytic and geometric properties of the SLE processes depend crucially on the value of the parameter κ which determines the intensity of the forcing—larger values of κ correspond to less regularity.

The author shows that a precise formulation of the conformal invariance conjecture for LERW implies that its scaling limit is described by SLE, with $\kappa = 2$. The main idea of the proof is to deform the unit disc into the complement of LERW starting at the center of the disc, using the solution of Loewner's differential equation for some unknown driving process. The driving process is stationary and has the Markov property, and hence must be a Brownian motion. The value of κ is identified by computing the twisting constant of LERW.

From MathSciNet, October 2008

Almut Burchard

MR1879850 (2002m:60159a) 60J65; 30C35, 82B41

Lawler, Gregory F.; Schramm, Oded; Werner, Wendelin

Values of Brownian intersection exponents. I. Half-plane exponents.

Acta Math. **187** (2001), *no.* 2, 237–273.

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MR1879851 (2002m:60159b) 60J65; 30C35; 82B41

Lawler, Gregory F.; Schramm, Oded; Werner, Wendelin

Values of Brownian intersection exponents. II. Plane exponents.

Acta Math. **187** (2001), *no.* 2, 275–308.

The main goal of the first paper is to establish rigorously conjectures made by B. Duplantier and K. H. Kwon [Phys. Rev. Lett. **61** (1988), 2514–2517] about the values of intersection exponents of Brownian motion in a half-plane. Write $\mathbb{H} = \{z \in \mathbb{C}\colon\ \mathrm{Im}(z) > 0\}$ for the upper half-plane and $B^1, \dots, B^p$ for p independent planar Brownian motions started from distinct points in $\mathbb{H}$. The main result of the paper states that when $t \to \infty$

$$(1)\quad \mathbb{P}[\forall i \neq j \in \{1, \dots, p\},\ B^i[0,t] \cap B^j[0,t] = \emptyset \\ \text{and } B^i[0,t] \subset \mathbb{H}] = t^{-\tilde{\zeta}_p + o(1)},$$

with $\tilde{\zeta}_p = \frac{1}{6}p(2p+1)$. This formula is furthermore extended to the case of p packets of $n_1, \dots, n_p$ independent Brownian motions in the half-plane and also generalized for non-integer values of $n_1, \dots, n_p$.

These results are closely connected to questions about the dimension of the planar Brownian frontier and the existence of critical exponents for a wide class of two-dimensional systems from statistical physics, including percolation and self-avoiding walk. The link with percolation is discussed at the end of the first paper.

The main idea used to establish (1) is a so-called "universality" idea developed by Lawler and Werner [J. Eur. Math. Soc. (JEMS) **2** (2000), no. 4, 291–328; MR1796962 (2002g:60123)], involving Brownian excursions and the SLE_6 process introduced by Schramm [Israel J. Math. **118** (2000), 221–288; MR1776084 (2001m:60227)]. The SLE_6 process is a conformal deformation of the half-plane $\mathbb{H}$ (or of any simply-connected domain) into a subset which shrinks with time. It is described in terms of K. Löwner's differential equation [Math. Ann. **89** (1923), 103–121; JFM 49.0714.01] with driving process a Brownian motion rescaled by $\sqrt{6}$. The key steps of the paper are to prove a "locality" property for the SLE_6 process and then to obtain a generalized Cardy's formula.

In the second paper, the authors focus on plane exponents. Their main result states that the probability for n independent Brownian motions started from n different points not to intersect each other up to time t equals $t^{-\zeta_n + o(1)}$, where $o(1)$ tends to 0 when $t \to \infty$ and $\zeta_n = \frac{1}{24}(4n^2 - 1)$. This formula is furthermore generalized for k packets of $n_1, \dots, n_k$ planar Brownian motions, and also for non-integer values of $n_1, \dots, n_k$.

An interesting consequence of these formulas is the determination of the Hausdorff dimension of some exceptional subsets of a planar Brownian path. For example, using a theorem of theirs in ["Analyticity of intersection exponents for planar Brownian motion", Acta. Math., to appear], the authors prove B. B. Mandelbrot's conjecture [*The fractal geometry of nature*, Freeman, San Francisco, Calif., 1982; MR0665254 (84h:00021)], which claims that the Hausdorff dimension of the Brownian frontier is 4/3 almost surely. The Hausdorff dimension of cut points or pioneer points of a planar Brownian path are also derived.

As in the first paper, the main idea to compute the values of the planar Brownian intersection exponents is to use a "universality" idea involving stochastic Löwner evolution processes SLE_6 and two-dimensional Brownian excursions. It has to be

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mentioned that SLE_6 processes are conjectured to correspond to the scaling limit of two-dimensional critical percolation cluster boundaries, so this work is closely related to important questions in theoretical physics.

The main tools of the proofs are conformal mapping theory (including the Schwartz lemma and the Koebe $\frac{1}{4}$-theorem), stochastic calculus and planar Brownian excursions theory.

From MathSciNet, October 2008

Christophe Giraud

MR1992830 (2004g:60130) 60K35; 60J65, 82B27

Lawler, Gregory; Schramm, Oded; Werner, Wendelin

Conformal restriction: the chordal case.

J. Amer. Math. Soc. **16** (2003), *no.* 4, 917–955.

The SLE_κ processes are obtained in solving Loewner's differential equation with driving term $B(\kappa t)$, with B a Brownian motion and $\kappa > 0$. It is believed they appear in the scaling limit of many two-dimensional models in statistical physics for which conformal field theory has been applied. This has been proved for some of them, including site-percolation on the triangular lattice, loop-erased random walks and the uniform spanning tree Peano path.

Recently, the authors used a restriction property to relate the SLE_6 process to the planar Brownian motion and compute in this way some "intersection exponents" for Brownian paths [G. F. Lawler, O. Schramm and W. Werner, Acta Math. **187** (2001), no. 2, 237–273; MR1879850 (2002m:60159a); Acta Math. **187** (2001), no. 2, 275–308; MR1879851 (2002m:60159b); V. N. Kolokol′tsov and A. E. Tyukov, Markov Process. Related Fields **7** (2001), no. 4, 603–625; MR1893144 (2002m:60163); G. F. Lawler, O. Schramm and W. Werner, Acta Math. **189** (2002), no. 2, 179–201; MR1961197 (2003m:60231)]. In the present paper, they investigate the probability distributions on the closed subsets of the upper half-plane H, satisfying the so-called conformal restriction property. A random subset K is said to satisfy this property if (very roughly and when it makes sense) for $D \subset H$, the law of K conditioned on $K \subset D$ is equal to the law of $\phi(K)$, where ϕ is a conformal map from H onto D, preserving 0 and ∞.

The authors show there exists a one-parameter family of probability measure satisfying the conformal restriction property, they characterize it and construct it from the SLE_κ processes for suitable κ. In addition, they prove that under one of these probability measures, a set K has a boundary of dimension $4/3$ and locally looks like an $SLE_{8/3}$. Finally, the law of the $SLE_{8/3}$ (which is expected to correspond to the scaling limit of the self-avoiding random walk) is shown to be the only law of this family of probability measures supported on simple curves.

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Christophe Giraud

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