BOOK REVIEWS

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Holomorphic Morse inequalities and Bergman kernels, by Xiaonan Ma and George Marinescu, Progress in Mathematics, 254, Birkhäuser Verlag, Basel, 2007, xiv+422 pp., US$79.95, ISBN 13: 978-3-7643-8096-0

Many problems in complex geometry involve the analysis of high powers $L^N \to M$ of an ample holomorphic line bundle over a complex manifold M of dimension m. When M is compact, there are no non-constant holomorphic functions on M and instead one uses holomorphic sections of L^N for $N = 1, 2, 3, \dots$ to do complex analysis. Examples of line bundles and holomorphic sections are given at the beginning of §1. The space $H^0(M, L^N)$ of all holomorphic sections of L^N is analogous to the space of (holomorphic) homogeneous polynomials of degree N and is equal to that space in the case $H^0(\mathbb{P}^m, \mathcal{O}(N))$, i.e., $M = \mathbb{P}^m$ (complex projective space) and $L = \mathcal{O}(1)$ (the hyperplane section bundle).

An important use of holomorphic sections of a line bundle L is to define holomorphic maps to projective space. A section can be expressed locally over an open set $U \subset M$ in the form $s = fe$, where f is a local holomorphic function ($f \in \mathcal{O}(U)$) and where e is a local frame, i.e., a local non-vanishing section over U. A collection $\{s_1, \dots, s_{k+1}\}$ of sections determines a local map $(f_0, \dots, f_k) : U \to \mathbb{C}^{k+1}$, and it is easy to see that the map changes only by a scalar multiple if the frame e is changed. Hence, one obtains a map $[f_0, \dots, f_k] : M \to \mathbb{P}^k$ to projective space (lines through 0 in $\mathbb{C}^{k+1}$). The map is often written as $[s_0, \dots, s_k]$. The extreme case is to use an entire basis $\mathcal{S}$ of sections to define the (Kodaira type) map

$$\Phi_{\mathcal{S}} = [s_0, \dots, s_{d_N}] : M \to \mathbb{P}^{d_N} \tag{1}$$

with $d_N + 1 = \dim H^0(M, L)$. The line bundle is called *very ample* if $\Phi_{\mathcal{S}}$ is an embedding. This property is independent of the choice of basis, since they differ by the action of $SL(d_N + 1, \mathbb{C})$ on $\mathbb{P}^{d_N}$.

Another important use of holomorphic sections is to define subvarieties of M by zero sets $Z_s = \{z : s(z) = 0\}$. One may construct more general subvarieties as fibers $[s_0, \dots, s_k]^{-1}(p)$ of maps to projective space or as simultaneous zeros $\{z : s_1(z) = \cdots = s_k(z) = 0\}$ of several sections.

Often it is difficult to know how such maps or zero sets behave for a fixed line bundle L. Additional control comes by taking tensor powers L^N. A line bundle is called *ample* if there exists a positive integer N so that L^N is very ample, i.e., if (1) is an embedding for a basis $\mathcal{S}$ of $H^0(M, L^N)$. In this case, for sufficiently large

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N, the dimension $d_N + 1 = \dim H^0(M, L^N)$ is given by a polynomial of the form $d_N + 1 = \frac{c_1(L)^m}{\pi^m m!} N^m + \cdots$, where $c_1(L)$ is the first Chern class of L. There is a curvature condition (called *positivity*) for ampleness: L is ample if there exists a Hermitian metric h on L with positive curvature $(1,1)$ form ω_h. In a local frame e, the metric may be written $h = e^{-\varphi}$ and then $\omega_h = \frac{i}{\pi}\partial\bar{\partial}\varphi$. When the line bundle fails to be ample, it is difficult to determine the dimension of $H^0(M, L^N)$. Asymptotic estimates on the dimensions of the qth cohomology group $H^q(M, L^N)$ of the sheaf of L^N-valued holomorphic sections are given by Demailly's holomorphic Morse inequalities [Dem2].

Holomorphic sections $s \in H^0(M, L^N)$ of high degree may be used to approximate general holomorphic objects. But the use of sections in $H^0(M, L^N)$ to make approximations has a much wider scope than approximating holomorphic objects, and it has become deeply embedded in complex geometry. A key use is to approximate general smooth metrics by polynomial type metrics known as Bergman (or Fubini-Study) metrics of degree (or height) N. The space $\mathcal{B}_N$ of Bergman metrics of degree N is by definition the finite-dimensional family of metrics obtained by pulling back the Fubini-Study metric on $\mathbb{P}^{d_N}$ by the maps (1) as $\mathcal{S}$ varies over all possible bases of $H^0(M, L^N)$. The program of Yau, Tian, Donaldson and others of relating *canonical metrics* on M to algebro-geometric properties of (M, L) is intimately connected to the use of the *algebro-geometric* Bergman metrics to approximate general *transcendental* Kähler metrics (cf. [D2, T, PS]). In fact, this approximation is closely related to the use of Bernstein polynomials to approximate general continuous functions [Z2].

Asymptotic analysis of holomorphic sections is important for other geometric problems as well. Donaldson defined and used *asymptotically holomorphic sections* on almost complex symplectic manifolds to construct embedded symplectic submanifolds [D1].

The book under review is the first to give a systematic book-length exposition of the applications of $H^0(M, L^N)$ to these and other problems in complex geometry and analysis on Kähler manifolds and, more generally, on almost-complex symplectic manifolds. It contains a fairly large and up-to-date bibliography drawing from a very wide variety of articles. It is written by geometric analysts from a PDE point of view, and it is devoted in large part to the asymptotic analysis of high powers of line bundles. The two main tools the book develops and uses are the Bergman/Szegö kernel and Demailly's holomorphic Morse inequalities.

The Bergman kernel $P_N(z, w)$ is given the most attention. It is the (Schwartz) kernel function of the orthogonal projection onto $H^0(M, L^N)$ with respect to an inner product induced by a Hermitian metric on L. Equivalently, it is the spectral projection for the eigenvalue 0 of the Kodaira Laplacian $\Delta_N = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$ on $\mathcal{L}^2(M, L^N)$. By the Bochner-Kodaira-Nakano formula, when (L, h) is a *positive* line bundle, 0 is isolated from the rest of the spectrum as N varies, and in fact the least non-zero eigenvalue of Δ_N drifts to the right like a constant times N. This spectral gap property allows for many constructions and approximations to P_N. For instance, the heat kernel $k_N(t, x, y)$ of Δ_N is a very good approximation in this case: one has $k_N(t, x, y) \to P_N(x, y)$ as $N \to \infty$ with a remainder that is $O(e^{-CNt})$. Thus, one can reduce calculations involving P_N to those involving $k_N(t, x, y)$ for large t, which are sometimes simpler or better known. Another option is to construct P_N as

$$P_N = \psi(\sqrt{\Delta}) = \int_{\mathbb{R}} \hat{\psi}(t)\cos(t\sqrt{\Delta_N})dt,$$

where ψ is a bump function equal to one near $\lambda = 0$ but which vanishes outside of a small enough interval so that no other eigenvalue of $\sqrt{\Delta_N}$ lies in its support. Then one can obtain information about P_N from knowledge of the wave equation associated to Δ_N. The authors give many detailed constructions and calculations of this kind, drawing from the literature on local index theory and in particular from work of J. M. Bismut. Scaling asymptotics of the heat kernel can also be used to prove the holomorphic Morse inequalities [B, Dem].

The Bergman/Szegö kernel may also be constructed directly as a complex oscillatory integral by a method due to Boutet de Monvel-Sjöstrand [BSj]. A new and very nice construction is given in [BBSj]. At least conceptually, this is a more direct approach than constructing the heat kernel or wave kernel and dropping the positive eigenvalue terms. It can also be used to study Morse inequalities [B1]. The Boutet de Monvel-Sjöstrand approach is however only briefly touched upon in the book under review.

We now give an overview of Bergman kernels and holomorphic Morse inequalities and their applications to a variety of problems in complex geometry.

1. Background on positive Hermitian holomorphic line bundles over Kähler manifolds

Let (M, ω) be a compact m-dimensional Kähler manifold. We recall that a Kähler manifold is a complex manifold possessing a Hermitian metric $ds^2 = \sum_{i,j=1}^m g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ whose associated $(1,1)$ Kähler form $\omega_g = \frac{i}{2}\sum_{i,j=1}^m g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is closed. We assume that $\frac{1}{\pi}\omega \in H^2(M, \mathbb{Z})$ so that there exists a holomorphic Hermitian line bundle $(L, h) \to M$ whose curvature $(1,1)$ form equals ω.

1.1. **Examples of ample line bundles and holomorphic sections.** Holomorphic line bundles and their holomorphic sections are well illustrated in the case of Riemann surfaces, i.e., complex curves of dimension $m = 1$. The line bundles have a different form depending on whether the genus g of the Riemann surface equals $0, 1$ or is ≥ 2.

When the genus $g = 0, M = \mathbb{P}^1$ is the Riemann sphere or one-dimensional projective space of complex lines through the origin in $\mathbb{C}^2$. It carries the *tautological* line bundle $\mathcal{O}(-1) \to \mathbb{P}^1$ whose line at a point $[z_0, z_1] \in \mathbb{P}^1$ is the line itself. The dual line bundle $\mathcal{O}(1)$ is the line of linear functionals on the lines of $\mathcal{O}(-1)$. Its square $\mathcal{O}(2)$ may be identified with the holomorphic tangent bundle $T^{(1,0)}\mathbb{P}^1$, whose holomorphic sections are holomorphic vector fields $f\frac{d}{dz}$. The line bundle $\mathcal{O}(1)$ is ample and its holomorphic sections may be identified with the space of homogeneous holomorphic polynomials of degree one on $\mathbb{C}^2$. The Nth tensor power $\mathcal{O}(N)$ is a line bundle whose holomorphic sections may be identified with the space of homogeneous holomorphic functions of degree N on $\mathbb{C}^2$. The identifications will be explained in §1.4.

When $g = 1$, then $M = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ is a lattice in $\mathbb{R}^2$. In this case, both the tangent and cotangent bundles are trivial, and the positive line bundles are powers of a different type of line bundle which we may call the *theta* line bundle $\Theta \to M$. Its sections are theta functions of level one, and in general $H^0(M, \Theta^N)$

consists of theta functions of level N. These sections lift to $\mathbb{C}$ as holomorphic functions $\theta(z,\tau)$ which transform under translations in Λ by so-called *multipliers*. When $N = 1$, $\theta(z+1,\tau) = \theta(z,\tau), \theta(z+\tau,\tau) = e^{-i\pi\tau}e^{-2\pi iz}\theta(z,\tau)$. The only periodic holomorphic functions with respect to Λ are constant functions, so this quasi-periodic function and its higher level analogues are the next best thing. One may express $\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{in^2\tau}e^{2\pi inz}$ as a one-dimensional Fourier series, and the higher level analogues are similar. Although they are complicated special functions, elements of $H^0(\mathbb{C}/\Lambda, \Theta^N)$ are, in many significant ways, no more complicated than polynomials of degree N. Indeed, the degree N of the line bundle is a measure of the degree of complexity of its sections. This theme is borne out by many of the results mentioned in this review, e.g., by the number and distribution of zeros of the sections.

When $g \geq 2$, a natural ample line bundle over the Riemann surface M is the canonical bundle $K = T^{*(1,0)}$, whose holomorphic sections are differentials fdz. Holomorphic sections of powers K^N are the holomorphic N-differentials $f(dz)^N$. As in the case of $\mathbb{C}/\Lambda$, one may uniformize by lifting the line bundle and its sections to the upper half-plane $\mathcal{H}$, and then the sections become holomorphic functions on $\mathcal{H}$ which transform by multipliers when translated by elements of the group $\Gamma \subset PSL(2,\mathbb{R})$ such that $\mathcal{H}/\Gamma = M$. The multiplier is determined by the fact that $f(dz)^N$ is Γ-invariant. The dual tangent bundle and its powers are negative when $g \geq 2$ and have no holomorphic sections.

1.2. **Positive line bundles.** The curvature form of a Hermitian line bundle (L,h) is defined by

$$\omega_h = -\frac{i}{2}\partial\bar{\partial}\log\|e_L\|_h^2\,, \tag{2}$$

where e_L denotes a local holomorphic frame (= non-vanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L,e_L)^{1/2}$ denotes the h-norm of e_L. The line bundle is said to be *positive* since it possesses a Hermitian metric with a Kähler curvature form. Positive line bundles are ample, which intuitively means that there exist sections in $H^0(M,L^N)$ with specified value and derivative at any point. The precise definition, given above, is that for some power L^N, the map (1) is an embedding. The dimension is given by the Riemann-Roch formula for large enough N and has the asymptotics $\dim H^0(M,L^N) = \frac{\mathrm{Vol}_\omega(M,g)}{\pi^m}N^m + O(N^{m-1})$. For this and related background, we refer to [GH]. We note that the book under review also covers non-compact Kähler manifolds.

More generally, complex geometers are interested in semi-positive lie bundles, big line bundles, pseudo-effective line bundles, and other weaker forms of positivity. A semi-positive line bundle is one possessing a Hermitian metric h for which ω_h is a semi-positive metric (i.e., its eigenvalues are ≥ 0 at each point). A big line bundle is one possessing a singular Hermitian metric whose curvature current is strictly positive. In these cases, $\dim H^0(M,L^N) \sim cN^m$, although the leading coefficient may be smaller than that of the Hilbert polynomial in the positive case. The book covers big line bundles in its Chapter 2. For another extensive exposition, see [Dem2].

1.3. **Bergman/Szegö kernels.** Bergman or Szegö kernels are orthogonal projections onto $H^0(M,L^N)$ with respect to an inner product. The most natural inner products are the ones induced by the Hermitian metric h. The metric h induces

Hermitian metrics h^N on L^N given by $\|s^{\otimes k}\|_{h^N} = \|s\|_h^N$. We then define the Hermitian inner product on sections by

$$\langle s_1, s_2\rangle_{h^N} = \int_M h^N(s_1, s_2)dV_h \qquad (s_1, s_2 \in H^0(M, L^N))\,. \tag{3}$$

Here, $dV_h = \frac{1}{m!}\omega_h^m$ is the volume form induced by the positive $(1,1)$ form ω_h. Since $\frac{1}{\pi}\omega$ is a de Rham representative of the Chern class $c_1(L) \in H^2(M, \mathbb{R})$, one has $\mathrm{Vol}(M) = \frac{\pi^m}{m!}c_1(L)^m$. The Szegö kernel (with respect to the Hermitian metric h) is the orthogonal projection $P_{h^N} : \mathcal{L}^2(M, L^N) \to H^0(M, L^N)$. In terms of an orthonormal basis $\{S_j^N\}$ of sections of $H^0(M, L^N)$, one has

$$P_{h^N}(z, w) = \sum_{j=1}^{d_N} S_j^N(z) \otimes \overline{S_j^N(w)}\,, \tag{4}$$

so that

$$(P_{h^N}s)(z) = \sum_{j=1}^{d_N} S_j^N(z) \int_M h_z^N\big(s(w), S_j^N(w)\big)_{h^N} dV_M(w)\,, \quad s \in \mathcal{L}^2(M, L^N)\,. \tag{5}$$

We note that in the book under review, the power is denoted p and the Bergman/-Szegö kernel is denoted P_p.

In this discussion, we have only considered the Bergman kernels on Kähler manifolds and line bundles. But the notion of an orthogonal projection onto the kernel of a differential (or pseudo-differential) operator acting on sections of a vector bundle is very general, and one uses the term *Bergman* kernel in this level of generality. In the book under review, Bergman kernels are studied both on Kähler manifolds and on almost Kähler symplectic manifolds.

1.4. **Lifting the Szegö kernel to the unit circle bundle.** To understand why P_{h^N} is called a Szegö kernel, we lift it to the unit circle bundle defined by the Hermitian metric. The book under review does not use this approach, but it is important to be aware of it. It originates in the work of H. Grauert and is discussed in some detail in [BG] and [BSZ], which we follow here.

We let L^* denote the dual line bundle to L, and we consider the circle bundle $X = \{\lambda \in L^* : \|\lambda\|_{h^*} = 1\}$, where h^* is the norm on L^* dual to h. Let $\pi : X \to M$ denote the bundle map; if $v \in L_z$, then $\|v\|_h = |(\lambda, v)|$, $\lambda \in X_z = \pi^{-1}(z)$. Thus, X is the boundary of the disc bundle $D = \{\lambda \in L^* : \rho(\lambda) > 0\}$, where $\rho(\lambda) = 1 - \|\lambda\|_{h^*}^2$. It was observed by Grauert that positivity of ω_h implies that D is strictly pseudoconvex in L^*. Hence, X is a strictly pseudoconvex CR manifold. The contact form $\alpha = -i\partial\rho|_X = i\bar{\partial}\rho|_X$ and the volume form

$$dV_X = \frac{1}{(m)!}\alpha \wedge (d\alpha)^m = \alpha \wedge \pi^* dV_h \tag{6}$$

are associated to X.

The sections $s \in H^0(M, L^N)$ lift to equivariant CR functions on X. That is, we define the Hardy space $\mathcal{H}^2(X) \subset \mathcal{L}^2(X)$ of square-integrable CR functions on X, i.e., functions that are annihilated by the Cauchy-Riemann operator $\bar{\partial}_b$ which are in $\mathcal{L}^2$ with respect to the inner product

$$\langle F_1, F_2\rangle = \frac{1}{2\pi}\int_X F_1 \overline{F_2} dV_X\,, \quad F_1, F_2 \in \mathcal{L}^2(X)\,. \tag{7}$$

Equivalently, $\mathcal{H}^2(X)$ is the space of boundary values of holomorphic functions on D that are in $\mathcal{L}^2(X)$. We let $r_\theta x = e^{i\theta}x$ ($x \in X$) denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial\theta}$. The S^1 action on X commutes with $\bar\partial_b$; hence $\mathcal{H}^2(X) = \bigoplus_{N=0}^\infty \mathcal{H}^2_N(X)$, where $\mathcal{H}^2_N(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{iN\theta}F(x)\}$. A section s of L^k determines an equivariant function $\hat{s}$ on L^* by the rule

$$\hat{s}(\lambda) = \left(\lambda^{\otimes N}, s(z)\right), \quad \lambda \in L^*_z, \ z \in M,$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$. Henceforth, we restrict $\hat{s}$ to X and then the equivariance property takes the form $\hat{s}_N(r_\theta x) = e^{iN\theta}\hat{s}(x)$. The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^N)$ and $\mathcal{H}^2_N(X)$.

The Szegö kernel of complex analysis is the (Schwartz kernel of the) orthogonal projection $P : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$, i.e., onto boundary values of holomorphic functions in D. The Bergman kernel is the orthogonal projection onto holomorphic functions in D. In the asymptotic analysis we are especially interested in the equivariant Szegö kernels $P_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$, i.e.,

$$P_N F(x) = \int_X P_N(x,y)F(y)dV_X(y), \quad F \in \mathcal{L}^2(X), \tag{8}$$

which are the Fourier components of P with respect to the S^1 action. The asymptotics of P_N are dual under the Fourier transform to the singularity on the diagonal of $P(x,y) = \sum_{N=1}^\infty P_N(x,y)$, which was determined by C. Fefferman [F].

The advantages of lifting to X are that the lifted sections of all $H^0(M, L^N)$ are functions rather than sections, and more seriously that they lie in one fixed Hilbert space. Obtaining asymptotics is essentially Fourier analysis with respect to the S^1 action.

1.5. **Model examples.** A basic example is the hyperplane bundle $\mathcal{O}(1) \to \mathbb{CP}^m$, the dual of the tautological line bundle. In this case, the circle bundle X is the $2m+1$ sphere S^{2m+1}, which is the boundary of the unit ball $B^{2m+2} \subset \mathbb{C}^{m+1}$. A straightforward calculation shows that

$$P_N(x,y) = \sum_J \frac{(N+m)!}{\pi^m j_0! \cdots j_m!} x^J \bar{y}^J = \frac{(N+m)!}{\pi^m N!}\langle x, y\rangle^N. \tag{9}$$

Note that

$$P(x,y) = \sum_{N=1}^\infty P_N(x,y) = \frac{m!}{\pi^m}(1 - \langle x,y\rangle)^{-(m+1)},$$

which is the classical Szegö kernel for the $(m+1)$-ball.

The second example is the linear model $\mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$ for positive line bundles $L \to M$ over Kähler manifolds and their associated Szegö kernels. Its associated principal S^1 bundle $\mathbb{C}^m \times S^1 \to \mathbb{C}^m$ may be identified with the *reduced Heisenberg group* $\mathbf{H}^m_{\rm red} = \mathbb{C}^m \times S^1$ with group law

$$(\zeta, e^{it}) \cdot (\eta, e^{is}) = (\zeta + \eta, e^{i[t+s+\Im(\zeta\cdot\bar\eta)]}).$$

The metric on $L_{\mathbf{H}}$ with curvature $\Theta = \sum dz_q \wedge d\bar{z}_q$ is given by setting $h_{\mathbf{H}}(z) = e^{-|z|^2}$; i.e., $|f|_{h_{\mathbf{H}}} = |f|e^{-|z|^2/2}$. For $N = 1, 2, \ldots$, $\mathcal{H}^2_N \subset \mathcal{H}^2(\mathbf{H}^m_{\rm red})$ is the (infinite-dimensional) Hilbert space of square-integrable CR functions f such that $f \circ r_\theta =$

$e^{iN\theta}f$ as before. The Szegö kernel $P_N^{\mathbf{H}}(x,y)$ is the orthogonal projection to $\mathcal{H}_N^2$, given by

$$P_N^{\mathbf{H}}(x,y) = \frac{1}{\pi^m} N^m e^{iN(t-s)} e^{N(\zeta\cdot\bar{\eta} - \frac{1}{2}|\zeta|^2 - \frac{1}{2}|\eta|^2)}\,, \qquad x = (\zeta, t)\,,\ y = (\eta, s)\,. \tag{10}$$

Another infinite volume example is provided by the hyperbolic disc $\mathbf{D}$ (or plane). In this case, the relevant line bundles are powers $K^\lambda \to \mathbf{D}$ of the canonical bundle and the Hilbert space is

$$\mathcal{H}^2_\lambda(\mathbf{D}) = \{f \in \mathcal{O}(\mathbf{D}),\ \int_{\mathbf{D}} |f(z)|^2 (1-|z|^2)^{\lambda-2} dm(z)) < \infty\},$$

where dm is the Lebesgue measure. An orthonormal basis is given by

$$\{e_n = \sqrt{\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)}} z^n\}.$$

Here, $\mathcal{O}(\mathbf{D})$ is the space of holomorphic functions on $\mathbf{D}$. The Bergman kernel for this space is

$$B_\lambda(z,w) = (1 - z\bar{w})^{-\lambda} = \sum_{m=0}^{\infty} (\lambda)_m \frac{(z\bar{w})^m}{m!},$$

where

$$(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1).$$

1.6. **Kähler quantization and the Boutet de Monvel-Sjöstrand parametrix.** The approach we are describing is that of Kähler quantization. Geometric (and in particular Kähler) quantization links symplectic geometry (manifolds, functions and maps) to Hilbert space analysis (Hilbert spaces, Toeplitz operators, and Fourier integral operators). Thus, the Hardy space $\mathcal{H}^2(X)$ is the quantization of the symplectic cone $\Sigma = \{(x, r\alpha_x) : r > 0, x \in X\} \subset T^*X$ and the Bergman/Szegö kernel P is the quantization of the restriction to this cone. The spaces $H^0(M, L^N)$ both in the Kähler and symplectic settings arise from geometric quantization of (M, ω) and the parameter $\frac{1}{N}$ is the Planck's constant in this setting.

By analyzing the symplectic geometry of Σ and its quantization, Boutet de Monvel-Sjöstrand [BSj] constructed a parametrix for P, i.e., a complex oscillatory kernel

$$S(x,y) \sim \int_0^\infty e^{it\psi(x,y)} s(x,y,t) dt, \tag{11}$$

with the same singularity as P on the diagonal. Here, the amplitude is a *classical symbol* possessing the expansion $s \sim \sum_{n=0}^\infty t^{m-n} s_n(x,y)$ and the phase $t\psi(x,y)$ is of *positive type*, i.e., complex valued but with non-negative imaginary part. Their parametrix construction gave a new proof of C. Fefferman's results on the singularity on the diagonal of $P(x,y)$ [F].

One obtains the Fourier components by averaging,

$$P_N(x,y) \sim \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} e^{it\psi(x, r_\theta y)} s(x, r_\theta y, t) e^{-iN\theta} dt d\theta. \tag{12}$$

If one changes variables $t \to Nt$ and applies the stationary phase method, one can eliminate the integrals and obtain a simple asymptotic expression

$$P_N(z,0;w,0) = e^{N(\varphi(z,w) - \frac{1}{2}(\varphi(z)+\varphi(w)))} \chi(d(z,w))\ s_k(z,w) + O(N^{-\infty}), \tag{13}$$

where φ is a local Kähler potential for ω_h and $\varphi(z,w)$ is its (almost-) analytic extension. Also, $\chi(d(z,w))$ is a smooth cutoff to a neighborhood of the diagonal in $M \times M$. The real part of the phase, which controls the decay of the integral, is equal to the negative of the *Calabi diastasis function*

$$\Im t = D(z,w) := \varphi(z,w) + \varphi(w,z) - (\varphi(z) + \varphi(w)),$$

an intrinsic kind of distance between points determined by the Kähler form ω. It does not depend of the choice of Kähler potential for ω.

As mentioned above, the recent article of Berman-Berndtsson-Sjöstrand [BBSj] contains a very readable and novel construction of the parametrix in the form (13). It constructs the approximate Bergman/Szegö kernel in the form of a Fourier integral over a *good contour*. The proof that the approximate kernel truly approximates the true Bergman/Szegö kernel is deduced from the Hörmander $\bar{\partial}$ estimates.

1.7. **Diagonal and off-diagonal asymptotics.** There are many geometric applications of the asymptotics on and off the diagonal of the Szegö kernel $P_N(z,w)$. This parametrix was used by D. Catlin [C] and independently by the reviewer [Z] to obtain asymptotics on the diagonal of the Szegö kernel, and applications to Tian's almost isometry theorem [T]. In subsequent work with P. Bleher and B. Shiffman (cf. [SZ, BSZ]), off diagonal asymptotics on balls of radius $\frac{N^{1/6}}{\sqrt{N}}$ were determined. One way to state them is by rescaling the Szegö kernel in small balls and in special *Heisenberg coordinates*:

(14)
$$\begin{aligned} &N^{-m} P_N(P_0 + \tfrac{u}{\sqrt{N}}, \tfrac{\theta}{N}; P_0 + \tfrac{v}{\sqrt{N}}, \tfrac{\varphi}{N}) \\ &\quad = \Pi_1^{\mathbf{H}}(u,\theta;0,0)\left[1 + \textstyle\sum_{r=1}^{K} N^{-r/2} b_r(P_0,u,0)\right] + N^{-(K+1)/2} R_K(P_0,u,N)\,, \end{aligned}$$

where

$$\|R_K(P_0,u,N)\|_{\mathcal{C}^j(\{|u|\le N^{1/6}\})} \lesssim C_{K,j} e^{-\frac{1-\epsilon}{2}|u|^2} \quad \text{for } j \ge 0\,, \tag{15}$$

and where $b_r(P_0,u,v)$ are certain polynomials. Thus, on small enough balls, the Szegö kernel is a small perturbation of the Heisenberg Szegö kernel of the same degree. On the diagonal one has

$$P_N(z,0;z,0) = a_0 N^m + a_1(z) N^{m-1} + a_2(z) N^{m-2} + \cdots \tag{16}$$

for certain smooth coefficients $a_j(z)$ with $a_0 = \pi^{-m}$. The next two coefficients were calculated by Z. Lu [Lu].

The off-diagonal expansion is covered in Chapter 4 and is given a lot of attention in this book. They give off-diagonal expansions for P_N for various elliptic differential operators such as the Kodaira-Laplacian operator, the spin-$\mathbb{C}$ Dirac operator and others, and improve upon (14) by giving exponentially small remainders. They derive the off-diagonal asymptotics from those of the heat or wave kernels, rather than by the method above. An omission is that they do not seem to mention that off-diagonal asymptotics, in both the Kähler and almost Kähler case, were earlier derived in both the Kähler and almost Kähler case from the Boutet de Monvel-Sjöstrand parametrix (see [BSZ, SZ]). As with many applications in this book, the off-diagonal asymptotics are often more pertinent than the on-diagonal ones. For instance, as discussed below, they give a simple analytic proof of Kodaira embedding type theorems.

The expansions for line bundles admit several generalizations to holomorphic vector bundles E. One might consider twists $L^k \otimes E$ or symmetric powers $\mathrm{Sym}^k E$. In both cases, by enlarging M, one can reduce the existence of the expansion to the case of powers of line bundles. The heat kernel approach may be simpler for actual calculations. The calculation of the trace asymptotics for $L^k \otimes E$ was carried out in [W].

1.8. **Applications.** Bergman/Szegö kernels can be used in many applications. If we fix y and L^2 normalize, we obtain a *coherent state* $\Phi_N^y(x) = \frac{P_N(x,y)}{\sqrt{P_N(y,y)}}$ based at y. It corresponds to a holomorphic section of L^N which is *most peaked* at y. On small balls of radius $\frac{C}{\sqrt{N}}$, it is like a Gaussian bump of height $N^{\frac{m}{2}}$ centered at y and with width $\frac{C}{\sqrt{N}}$, although its tails are longer than Gaussian globally on the manifold. Since it is almost a delta function at y for large N, the map

$$y \to \Phi_N^y \tag{17}$$

can be proved to define an embedding $\Phi_N : X \to \mathcal{H}_N^2(X)$. This is an analytic version of the Kodaira embedding theorem. It is an equivariant lift to X of the usual Kodaira embedding (1).

Chapters 5–8 of the book under review contain many applications of the on- and off-diagonal asymptotics of the Szegö kernel to problems in geometry and analysis. Among them are the following.

- Relating stability of polarized line bundles to existence of constant scalar curvature Kähler metrics in the Kähler class $c_1(L)$. Stability is an algebraic-geometric notion, while scalar curvature metrics are transcendental differential geometric. A key link between algebraic geometry and differential geometry is provided by the space $\mathcal{B}_N$ of Bergman (= Fubini-Study) metrics, i.e., metrics obtained by pulling back the Fubini-Study metric under Kodaira maps given by any basis of $H^0(M, L^N)$. It follows from [T, C, Z] that when the bases are orthonormal with respect to $\langle , \rangle_{h^N}$, then the Bergman metrics approximate any Kähler metric in the given Kähler class in a very strong asymptotic sense. This approximation was used in [D2] to show that existence of a constant scalar curvature metric implies stability in a certain sense. A key point was that the second term a_1 in the expansion (16) is the scalar curvature s_ω of ω_h [Lu] . This term could also be obtained by some basic invariant theory (up to a constant, s_ω is the geometric invariant which could occur in this term). On a deeper level, $\mathcal{B}_N$ is a finite-dimensional symmetric space whose geometry approximates the infinite-dimensional symmetric space $\mathcal{H}$ of Kähler metrics in a fixed Kähler class. For instance, Phong-Sturm proved [PS2] that geodesics of $\mathcal{B}_N$ (which are one-parameter subgroups) approximate geodesics of $\mathcal{H}$ (which are solutions of a complex homogeneous Monge-Ampère equation). For further exposition in this direction, see [PS].
- Zeros of random holomorphic sections. Here one equips $H^0(M, L^N)$ with a Gaussian measure induced by h, and one considers the expected distribution of zeros. The two-point function is $P_{h^N}(z, w)$, and from the diagonal asymptotics one finds that the expected limit distribution of zeros of random sections is the curvature $(1,1)$-form (cf. [BSZ]). This is the equilibrium measure in this situation; R. Berman recently proved more general results in

this direction [B2]. P. Bleher, B. Shiffman, and the reviewer [BSZ, BSZ2] used the off-diagonal asymptotics (14) to obtain correlation functions as well, although the book does not pursue the subject that far.

- Asymptotics of the holomorphic analytic torsion. This is defined as a determinant of a certain Laplacian on $L^N \otimes E$, where E is a vector bundle. It is used to define the Quillen metric. Bismut-Vasserot studied the asymptotics as $N \to \infty$ of the torsion. Applications of the torsion to canonical metrics are discussed in [MW], which escaped the bibliography but is another useful reference.
- Geometry of almost-complex symplectic manifolds. There are several types of replacements for $H^0(M, L^N)$ when (M, ω, J) is an almost complex symplectic manifold with integral form ω and compatible complex structure J. Donaldson's paper [D1] initiated a new field of constructing analogues of Kähler asymptotic analysis in the symplectic setting. His purpose was to construct embedded symplectic submanifolds as zero sets of certain special asymptotically holomorphic sections of the *pre-quantum* line bundles $L^N \to M$ determined by ω. Motivated by his paper, Borthwick-Uribe [BU], Shiffman-Zelditch [SZ], R. Paoletti, and the authors of this book (among others) considered Bergman/Szegö kernel analysis for a variety of spaces of sections replacing $H^0(M, L^N)$. The space in [BU] is the space (first defined in [GU]) of *low-lying* eigensections for a certain modified $\Box_b$ Laplacian in the almost Kähler setting, which equals $\Box_b$ in the Kähler case. The space in [SZ] is a space of *almost holomorphic sections* defined by Boutet de Monvel-Guillemin [BG] where the non-integrable $\bar{\partial}_b$ operator is deformed by a pseudo-differential perturbation to an integrable operator satisfying $D_b^2 = 0$. The authors of this book also consider the spin-$\mathbb{C}$ Dirac operator and its kernel. All of these spaces of sections provide asymptotically holomorphic sections in the sense of [D1]. Kodaira embedding theorems were proved in [BU]; they were derived in [SZ] for almost holomorphic sections as a corollary of the off-diagonal expansion (14). The low-lying eigensections of [GU, BU] are those of a differential operator for which good estimates may be expected, while those of [BG, SZ] have a Bergman/Szegö kernel that is very similar to the Kähler case.

2. Holomorphic Morse inequalities

We now consider the second large topic of the book. Holomorphic Morse inequalities are used to estimate dimensions $h^q(L^N) := \dim\ H^q(M, L^N)$ of holomorphic sections of line bundles which are not necessarily positive. In this generality, one has no Boutet de Monvel-Sjöstrand parametrix, and there are few alternatives to the heat or wave kernel approximation. (One alternative can be found in the recent article of R. Berman [B2].)

The holomorphic Morse inequalities of J. P. Demailly [Dem] assert that

$$\dim H^q(M, L^N) \leq (-1)^q \frac{N^n}{n!} \int_{M(q)} (\frac{i}{2\pi} c_1(L))^n + o(N^n). \tag{18}$$

Here, $M(q) \subset M$ is the subset in which $ic_1(L)$ has precisely q negative eigenvalues. It was observed by J. M. Bismut [B] that the leading order scaling asymptotics of the heat kernel could be used to simplify the proof of these inequalities. Bismut's proof was substantially simplified by J. P. Demailly [Dem] and by T. Bouche. We denote by $\bar{\Box}_N^q = (\bar{\partial} + \bar{\partial}^*)^2$ the complex Laplacians on $(0,q)$ forms with values in L^N. The heat operator is given by $e^{-\bar{\Box}_N^q}$ and we denote its kernel by $k_N^q(t,x,y)$. We define the time-scaled diagonal heat kernel by $e_N^q(t,x) := k_N^q(t/N,x,x)$. Then

$$e_N^q(t,x) \sim e_{\infty 0}^q(t,x) := (\sum_{|J|=q} e^{t\tilde{\alpha}_J}) \prod_{j=1}^n \frac{t\alpha_j(x)}{\sinh t\alpha_j(x)},$$

where $\tilde{\alpha}_J = \sum_{j \notin J} \alpha_j - \sum_{j \in J} \alpha_j$. Since one obviously has

$$\dim H^q(M, L^N) \leq \int_M k_N^q(t/N, x, x) dV \tag{19}$$

for any t, we can let $N \to \infty$ to obtain

$$\lim_{N\to\infty} N^{-n} \dim H^q(M, L^N) \leq \int_M t^{-n} e_{\infty o}^q(t,x) dV \tag{20}$$

and then let $t \to \infty$ to obtain

$$\lim_{N\to\infty} N^{-n} \dim H^q(M, L^N) \leq (-1)^q \frac{1}{n!} \int_{M(q)} (\frac{i}{2\pi} c_1(L))^n. \tag{21}$$

In the last step we used that (as $t \to \infty$),

$$\frac{e^{t\alpha} t\alpha}{\sinh t\alpha} = \frac{t\alpha}{1 - e^{-2t\alpha}} \sim \begin{cases} t\alpha + O(te^{-2\alpha t}), & \alpha > 0, \\ 1, & \alpha = 0, \\ O(te^{-2|\alpha| t}), & \alpha < 0. \end{cases} \tag{22}$$

In the case of $q = 0$, we have

$$e_{\infty 0}^0(t,x) \sim \begin{cases} t^r \prod_{i=1}^r \alpha_i(x), & \alpha_i(x) \geq 0 \ \forall i, \ \alpha_i(x) > 0 \ i = 1, \ldots, r, \\ 1, & \alpha_i(x) = 0 \ \forall i, \\ 0, & \exists i : \alpha_i(x) < 0. \end{cases} \tag{23}$$

For general q the asymptotics depend in a more complicated way on the eigenvalues of $ic_1(L)$. Assume for simplicity that $ic_1(L)$ is non-degenerate at $x \in M(q)$, and let $J_-(x)$ denote the set of q indices for which $\alpha_j(x) < 0$ and, respectively, $J_+(x)$ denote the set of indices for which $\alpha_j(x) > 0$. The only term in $\tilde{\alpha}_J$ which makes a non-trivial asymptotic contribution is the one for which $J = J_-(x)$. Hence

$$\begin{aligned} t^{-n} e_{\infty 0}^q(t,x) &\sim t^{-n} \prod_j e^{t|\alpha_j(x)|} \tfrac{t\alpha_j(x)}{\sinh t\alpha_j(x)} \\ &\sim (-1)^q \prod_j \alpha_j(x). \end{aligned} \tag{24}$$

2.1. **Applications.** The holomorphic Morse inequalities were first proved by J. P. Demailly for application to the Grauert-Riemenschneider conjecture that X is Moishezon if it carries a semi-positive line bundle (L,h) which is positive at least at one point. The book reviews the proof and also gives further applications to non-positive line bundles. For instance, they prove the Ji-Shiffman theorem [JS] that Moishezon manifolds are those that admit a strictly positive singular polarization (L,h) (i.e., a singular metric whose curvature current is strictly positive). There are related theorems of Bonavero and Takayama. They also discuss Boucksom's theorem on volumes of pseudo-effective line bundles. R. Berman has recently proved local versions of the holomorphic Morse inequalities directly using the Bergman kernel for non-positively curved metrics [B1, B2].

3. Final remarks

The book under review is very thorough and quite successful in providing a unified approach to a large and diverse collection of results on the asymptotic analysis of line bundles, most of which are not discussed in any other book. In addition, it contains Problem sections at the end of each chapter. It could be used as a textbook on the subject in a graduate level class.

As mentioned above, the book takes a consistent *heat kernel* (or at times *wave kernel*) approach to Bergman kernels and holomorphic Morse inequalities. This makes sense, since one has a parametrix for the heat kernel whether or not ω is a positive $(1,1)$ form, and so it is very useful for both of the main topics of the book. Moreover, heat kernel expansions are often easier to compute than Szegö kernel expansions. The reviewer has already indicated that the microlocal or *Kähler quantization* aspects are omitted. Aside from that, the book under review provides a very detailed and thorough introduction to asymptotics of line bundles and holomorphic sections, which takes the reader from the basic definitions to many of the research problems of contemporary interest.

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