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Spherical CR geometry and Dehn surgery, by Richard Evan Schwartz, Princeton University Press, 2007, 186 pp., hardback, ISBN-13: 978-0-691-12809-2, paperback, ISBN-13: 978-0-691-12810-8

1. Introduction

The ideas in this book revolve around a central theorem, the *horotube surgery theorem*, which is mainly of interest to specialists in complex hyperbolic geometry; see Section 6 below. This book is the culmination of several important papers by Schwartz in this area, which we describe in Section 4 below. As well as the main theorem, this book contains background material and applications. It may be used as an introduction to Schwartz's contribution to complex hyperbolic geometry as well as to the subject as a whole, and it is accessible to non-specialists. In this review we will give some of the historical background, discuss the horotube surgery theorem, and describe its applications. This plan follows the book and touches on several themes it contains.

2. Complex hyperbolic space

There are several ways to generalise the hyperbolic plane and its isometry group to objects in higher dimensions. Perhaps the most familiar is (real) hyperbolic three space, popularised by the work of Thurston [14]. The Poincaré disc and half plane models of the hyperbolic plane naturally come with a complex structure and it is natural to generalise them to complex hyperbolic space in higher complex dimensions; see [4] or [8] for further details. A useful model for complex hyperbolic space is the unit ball in $\mathbb{C}^n$ equipped with the Bergman metric. When $n = 1$, this is just the Poincaré metric on the unit disc in $\mathbb{C}$. When $n \geq 2$, complex hyperbolic space does not have constant curvature but has pinched negative curvature, which we normalise to lie between -1 and $-1/4$.

From now on we concentrate on the case $n = 2$. The hyperbolic plane is isometrically embedded into complex hyperbolic two-space $\mathbf{H}^2_{\mathbb{C}}$ in two geometrically distinct ways. First, the intersection of the unit ball in $\mathbb{C}^2$ with a complex line (for example one of the complex coordinate axes) is a totally geodesic disc. The restriction of the Bergman metric to this disc is the Poincaré metric with constant curvature -1. On the other hand, the intersection of $\mathbf{H}^2_{\mathbb{C}}$ with a Lagrangian plane (for example the collection of points with real coordinates) is also a totally geodesic disc. In this case, the restriction of the Bergman metric is the Klein metric on the hyperbolic plane with constant curvature $-1/4$.

The group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ is the projective unitary group $\mathrm{PU}(2,1)$. It is often useful to lift to the matrix group $\mathrm{SU}(2,1)$, which is a three-fold cover of $\mathrm{PU}(2,1)$. Non-trivial elements of $\mathrm{PU}(2,1)$ fall into the three classes familiar from real hyperbolic geometry. Namely, $A \in \mathrm{PU}(2,1)$ is *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$, one of which is attractive and the other repulsive; A is *parabolic* if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$ and is *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$. Elliptic isometries are either a *complex reflection* fixing a point or a complex line, or else they are called *regular*. Complex reflections correspond

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to matrices in $\mathrm{SU}(2,1)$ with a repeated eigenvalue and regular elliptic maps correspond to matrices with distinct eigenvalues. The full group of complex hyperbolic isometries $\widehat{\mathrm{PU}}(2,1)$ is generated by $\mathrm{PU}(2,1)$ and an antiholomorphic reflection fixing a Lagrangian plane. An example of such an involution is complex conjugation of both coordinates, which fixes the Lagrangian plane with real coordinates. Furthermore, any element of $\mathrm{PU}(2,1)$ may be written as the product of two reflections in Lagrangian planes [3].

The natural geometry associated to the boundary of real hyperbolic space is conformal geometry. Thus the boundary of a hyperbolic 3-manifold or orbifold naturally carries a conformal structure. In just the same way, the natural geometry associated to $\partial\mathbf{H}^2_{\mathbb{C}}$ is *spherical CR geometry* and the boundary of a complex hyperbolic 2-manifold or orbifold carries a spherical CR structure. For example, Schwartz has constructed a complex hyperbolic orbifold whose boundary is the Whitehead link complement, which therefore carries a spherical CR structure; see Theorem 4.3 below.

Three points of $\partial\mathbf{H}^2_{\mathbb{C}}$ are completely determined up to $\mathrm{PU}(2,1)$ equivalence by Cartan's angular invariant $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3) \in [-\pi/2, \pi/2]$. This invariant measures how the triple z_1, z_2, z_3 is aligned relative to the complex structure in the following sense. Denote the complex line spanned by z_1 and z_2 by L_{12}. Let Π_{12} be orthogonal projection onto L_{12}. Consider the triangle in L_{12} with vertices z_1, z_2, $\Pi_{12}(z_3)$. The angular invariant $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3)$ is half the signed area of this triangle with respect to the natural Poincaré metric on L_{12}. Hence if $z_3 \in L_{12}$, this triangle is ideal and has area $\pm\pi$, the sign depending on whether moving around the boundary from z_1 we meet the vertices in the order z_1, z_2, z_3 or in the order z_1, z_3, z_2. Thus, in this case the angular invariant is $\mathbb{A} = \pm\pi/2$. On the other hand, if z_1, z_2, z_3 lie in a Lagrangian plane, then $\Pi_{12}(z_3)$ lies on the geodesic with endpoints z_1 and z_2. In this case the triangle is degenerate and has area 0. Thus the angular invariant is also $\mathbb{A} = 0$.

3. Triangle groups

A *triangle group* Δ is the group generated by reflections in the side of a triangle. If the internal angles of the triangle are π/p, π/q, π/r, then $\Delta = \Delta(p,q,r)$ has the presentation

$$\Delta = \Big\langle \iota_1,\, \iota_2,\, \iota_3 \;:\; \iota_1^2 = \iota_2^2 = \iota_3^2 = (\iota_1\iota_2)^p = (\iota_2\iota_3)^q = (\iota_3\iota_1)^r = 1 \Big\rangle.$$

It is often useful to speak of the index 2 subgroup of Δ comprising products of even numbers of reflections, which we denote by Δ^+. Writing $\iota_1\iota_2 = \alpha$ and $\iota_2\iota_3 = \beta$, the group $\Delta^+ = \Delta^+(p,q,r)$ has presentation

$$\Delta^+ = \Big\langle \alpha,\, \beta \;:\; \alpha^p = \beta^q = (\alpha\beta)^r = 1 \Big\rangle.$$

The groups Δ and Δ^+ have faithful representations to the isometry group of the sphere, the Euclidean plane, or the hyperbolic plane depending on whether $1/p + 1/q + 1/r - 1$ is positive, zero, or negative respectively. In the hyperbolic case the internal angles of the triangle may be zero. In this case we allow p, q, or r to be infinity, and we remove the corresponding relation from each of the above presentations. In particular, $\Delta(\infty,\infty,\infty)$ is the free product of three groups of order 2, and $\Delta^+(\infty,\infty,\infty)$ is a free group on two generators.

In what follows, we restrict our attention to the hyperbolic case; that is, we suppose $1/p + 1/q + 1/r < 1$. For such p, q, r there is a triangle in the hyperbolic plane with internal angles π/p, π/q, π/r. Moreover, up to applying hyperbolic isometries, this triangle is unique. The group generated by reflections in the sides of this triangle is a faithful representation of $\Delta(p,q,r)$. This representation ρ is unique up to conjugacy. In higher dimensional (real) hyperbolic spaces, since there is a totally geodesic copy of the hyperbolic plane containing the three vertices of the triangle, the representation ρ is again unique up to conjugation. In contrast, this is not true in complex hyperbolic space.

Consider three complex lines L_1, L_2, and L_3 in $\mathbf{H}^2_{\mathbb{C}}$ for which the complex angle between L_1 and L_2 is π/p, the complex angle between L_2 and L_3 is π/q, and the complex angle between L_3 and L_1 is π/r. If I_j for $j = 1, 2, 3$ denotes the complex reflection of order 2 fixing L_j, then $\langle I_1, I_2, I_3\rangle$ is a representation of $\Delta(p,q,r)$. In contrast to the real hyperbolic case, the lines L_1, L_2, L_3 are not specified up to conjugation by the three angles π/p, π/q, π/r. In fact there is one more degree of freedom. This means that there is a one parameter family of representations of $\Delta(p,q,r)$.

In the special case when $p = q = r$, we can define an automorphism of $\Delta(p,p,p)$ that cyclically permutes ι_1, ι_2, and ι_3. A representation $\rho : \Delta(p,p,p) \longrightarrow \mathrm{PU}(2,1)$ is called *symmetric* if this automorphism is represented by an isometry J. Such a $J \in \mathrm{PU}(2,1)$ must have order 3 and satisfies $L_2 = J(L_1)$ and $L_3 = J^{-1}(L_1)$. This means that $I_2 = JI_1J^{-1}$ and $I_3 = J^{-1}I_1J$ and so $\langle I_1, I_2, I_3\rangle$ is an index 3 normal subgroup of $\langle I_1, J\rangle$.

4. Ideal triangle groups

An ideal triangle is one where all the interior angles are 0. The corresponding group is $\Delta(\infty,\infty,\infty)$. In this case a representation to $\mathrm{PU}(2,1)$ is generated by reflections of order 2 fixing complex lines L_1, L_2, L_3 which are pairwise asymptotic. Let $z_j = \partial L_{j-1} \cap \partial L_{j+1} \in \partial\mathbf{H}^2_{\mathbb{C}}$ with indices taken mod 3. The triple z_1, z_2, z_3 is determined up to $\mathrm{PU}(2,1)$ equivalence by the angular invariant $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3)$. Furthermore, we claim that $\rho : \Delta(\infty,\infty,\infty) \longrightarrow \mathrm{PU}(2,1)$ is determined up to conjugation by $\mathbb{A}$. In order to see this, choose three points z_1, z_2, and z_3 in $\partial\mathbf{H}^2_{\mathbb{C}}$ with angular invariant $\mathbb{A}$. Each pair of these points lies on a unique complex line, and so the z_1, z_2, z_3 completely determine three complex lines L_1, L_2, and L_3 and also determine the group $\langle I_1, I_2, I_2\rangle$ generated by order 2 complex reflections fixing these complex lines. Moreover, for any triple of points z_1, z_2, z_3 in $\partial\mathbf{H}^2_{\mathbb{C}}$ there exists J in $\mathrm{PU}(2,1)$ of order 3 satisfying $z_2 = J(z_1)$ and $z_3 = J^{-1}(z_1)$. Therefore the representation ρ is automatically symmetric.

We may then ask for which values of $\mathbb{A}$ the representation ρ is discrete and faithful. For example, when $\mathbb{A} = 0$, all three points lie on a Lagrangian plane. The intersections of L_1, L_2, and L_3 with this plane are geodesics and ρ is a Fuchsian representation preserving this Lagrangian plane. Hence ρ is discrete and faithful. On the other hand, when $\mathbb{A} = \pm\pi/2$, all three points lie on the same complex line, and so $I_1 = I_2 = I_3$ and the image of ρ is a group of order 2. This is certainly not faithful!

This question was investigated by Goldman and Parker [5] who proved the following theorem.

Theorem 4.1 ([5]). *Let* $\rho : \Delta(\infty,\infty,\infty) \longrightarrow \mathrm{PU}(2,1)$ *be a representation of an ideal triangle group for which all three generators are represented by complex involutions fixing complex lines. Suppose the three vertices have angular invariant* $\mathbb{A}$.

(i) *If* $\tan^2(\mathbb{A}) \leq 35$, *then* ρ *is discrete and faithful.*
(ii) *If* $\tan^2(\mathbb{A}) > 125/3$, *then* ρ *is either not discrete or not faithful. In particular, if* $\tan^2(\mathbb{A}) = \infty$, *then* ρ *is not faithful.*

Furthermore, Goldman and Parker conjectured that the condition in Theorem 4.1(ii) is necessary and sufficient. In [9] Schwartz gave a proof of this conjecture that depended on numerical analysis. Later, he gave a more conceptual proof in [12]. The main result of these papers may be summarised by

Theorem 4.2 ([9], [12]). *Let* $\rho : \Delta(\infty,\infty,\infty) \longrightarrow \mathrm{PU}(2,1)$ *be a representation for which all three generators are represented by complex involutions fixing complex lines. Then* ρ *is discrete and faithful if and only if* $\rho(\iota_1\iota_2\iota_3)$ *is loxodromic or parabolic. In particular, suppose that the three vertices have angular invariant* $\mathbb{A}$.

(i) *If* $\tan^2(\mathbb{A}) \leq 125/3$, *then* ρ *is discrete and faithful.*
(ii) *If* $125/3 < \tan^2(\mathbb{A}) < \infty$, *then* ρ *is not discrete.*
(iii) *If* $\tan^2(\mathbb{A}) = \infty$, *then* ρ *is not faithful.*

Subsequently, Schwartz investigated the geometry of the representation with $\tan^2(\mathbb{A}) = 125/3$, the last representation that is discrete and faithful. This group is sometimes called the *last ideal triangle group* or the *golden triangle group*. The geometry of this group is discussed in [10] and in Chapters 20 and 21 of the book under review.

Theorem 4.3 ([10]). *Let* $\rho : \Delta(\infty,\infty,\infty) \longrightarrow \mathrm{PU}(2,1)$ *be a representation of an ideal triangle group for which* $\rho(\iota_1\iota_2\iota_3)$ *is parabolic (that is, the three vertices have angular invariant* $\mathbb{A}$ *where* $\tan^2(\mathbb{A}) = 125/3$*). Let* $J \in \mathrm{PU}(2,1)$ *be the order* 3 *symmetry cyclically permuting the vertices. Then* $\rho(\Delta)$ *is discrete and faithful, the parabolic elements of* $\rho(\Delta)$ *are conjugate to powers of* I_jI_{j+1} *or* $I_1I_2I_3$, *and every other non-trivial element of* $\rho(\Delta)$ *is loxodromic.*

Moreover, if Ω *is the domain of discontinuity of* $\rho(\Delta)$, *then* $\Omega/\langle J, I_1JI_1\rangle$ *is the complement of the Whitehead link, the two components of the link corresponding to the parabolic conjugacy classes* $(I_1JI_1)J^{-1}$ *and* $(I_1JI_1)J$ *(that is, to* I_1I_2 *and* $I_1I_2I_3 = (I_1J)^3$, *respectively).*

This construction is the first example of a spherical CR structure being put onto a hyperbolic 3-manifold. It provides a bridge between complex hyperbolic Kleinian groups and the classical theory in hyperbolic 3-space. This bridge is the main philosophical starting point for the book under review. The *hyperbolic Dehn surgery theorem* of Thurston [14] is the main inspiration behind this book. The starting point of the hyperbolic Dehn surgery theorem is a cusped hyperbolic 3-manifold, such as a knot or link complement. A Dehn surgery is a recipe for capping off one of the cusps by gluing in a solid torus. Of course there are many ways to do this. The hyperbolic Dehn surgery theorem says that for all but finitely many Dehn surgeries, the resulting manifold is still hyperbolic.

Schwartz's goal is to take a cusped hyperbolic 3-manifold with a spherical CR structure and then to perform a Dehn surgery on one or more cusps to obtain new hyperbolic 3-manifolds with spherical CR structures. Before we discuss this result, we give a connection to other types of triangle groups.

5. Lagrangian triangle groups

In addition to the complex triangle groups discussed in the previous section, there is another type of representation of $\Delta(p,q,r)$ to the isometry group of complex hyperbolic space. Namely, we suppose that $\rho(\Delta) \in \widehat{\mathrm{PU}}(2,1)$ and each of the generators is represented by an antiholomorphic involution fixing a Lagrangian plane. In this case the product of two of the generators is represented by an elliptic or parabolic element of $\mathrm{PU}(2,1)$. Since any element of $\mathrm{PU}(2,1)$ can be written as the product of reflections in a pair of Lagrangian planes that intersect in $\mathbf{H}^2_{\mathbb{C}}$ (see [3]), there are no restrictions on the type of elliptic or parabolic maps that can occur in such a representation. This leads to more possible types of representation.

The only triangle group for which the Lagrangian representation space has been completely described is $\Delta(2,3,\infty)$, the index 2 extension of the classical modular group. This was done by Falbel and Parker in [2] and uses earlier work of Gusevskii and Parker [6] and Falbel and Koseleff [1]. There are different components of this representation space depending on whether the order 2 and order 3 generators of $\Delta^+(2,3,\infty)$ are represented by complex reflections in points or lines or by regular elliptic maps. In particular, for the order 2 generator $\alpha = \iota_1\iota_2$ we see that $A = \rho(\alpha) \in \mathrm{PU}(2,1)$ satisfies $A^2 = I$. The only possibilities are that A is a complex reflection fixing a point or a complex line. On the other hand, the order 3 generator $\beta = \iota_2\iota_3$ may be represented by a complex reflection in a point, a complex reflection in a line, or a regular elliptic map.

There is a copy of $\Delta(\infty,\infty,\infty)$ lying in $\Delta(2,3,\infty)$ as an index 6 subgroup. Once again we let $\mathbb{A}$ denote the angular invariant of the three parabolic fixed points of these generators. The main idea may be summarised as

Theorem 5.1 ([2]). *Let $\rho : \Delta(2,3,\infty) \longrightarrow \widehat{\mathrm{PU}}(2,1)$ be a representation for which all three generators are represented by antiholomorphic involutions fixing Lagrangian planes. Then ρ is discrete and faithful if and only if $\rho\big((\iota_1\iota_2\iota_3)^2\big)$ is loxodromic or parabolic. In particular, suppose that $A = \rho(\alpha) = \rho(\iota_1\iota_2)$ and $B = \rho(\beta) = \rho(\iota_2\iota_3)$ are the holomorphic elliptic maps of orders 2 and 3, respectively. Then we have the following.*

- (i) *If B is a complex reflection, then ρ is unique up to conjugacy and preserves a complex line. There are four such representations depending on whether A and B fix a point or a complex line.*
- (ii) *If A fixes a point and B is regular elliptic, then the representation is parametrised up to conjugacy by the angular invariant $\mathbb{A}$ of the three parabolic fixed points corresponding to AB, BA, and $B^{-1}AB^{-1}$. The representation is discrete and faithful for all $\mathbb{A} \in [-\pi/2, \pi/2]$.*
- (iii) *If A fixes a complex line and B is regular elliptic, then the representation is parametrised up to conjugacy by the angular invariant $\mathbb{A}$ of the three parabolic fixed points corresponding to AB, BA, and $B^{-1}AB^{-1}$. The representation is discrete and faithful if and only if $\tan^2(\mathbb{A}) \geq 15$.*

In the group from Theorem 5.1(iii) with $\tan^2(\mathbb{A}) = 15$ the element $\rho\big((\iota_1\iota_2\iota_3)^2\big)$ is parabolic. We can write this in terms of A and B as follows:

$$\rho\big((\iota_1\iota_2\iota_3)^2\big) = \rho(\iota_1\iota_2)\rho(\iota_3\iota_2)\rho(\iota_2\iota_1)\rho(\iota_2\iota_3) = [A, B^{-1}] = (AB)^{-1}[A,B](AB).$$

Remarkably, the group from Theorem 5.1(iii) with $\tan^2(\mathbb{A}) = 15$ is commensurable with the golden triangle group, that is, the group from Theorem 4.2 with

$\tan^2(\mathbb{A}) = 125/3$. We now explain this. Let $G_0 = \langle I_1, J\rangle$ be the index 3 normal extension of the golden triangle group. Hence, I_1 has order 2 and fixes a complex line, and J has order 3. Then $I_2 = JI_1J^{-1}$ and $I_3 = J^{-1}I_1J$. The parabolic elements of G_0 are conjugate to powers of $I_1I_2 = [I_1, J]$ and powers of I_1J (observe that $I_1I_2I_3 = (I_1J)^3$). Let $G_1 = \langle A, B\rangle$ be the group of words of even length in the group from Theorem 5.1(iii) with $\tan^2(\mathbb{A}) = 15$. Then A has order 2 and fixes a complex line, and B has order 3. The parabolic elements of G_1 are conjugate to powers of AB and powers of $[A, B]$. Thus we identify them by the map $\phi : G_0 \longrightarrow G_1$ by $\phi(I_1) = A$ and $\phi(J) = B$.

We may extend this identification to the other groups in Theorem 5.1(iii). For such groups $AB = \phi^{-1}(I_1J)$ is parabolic for all $\mathbb{A}$, but $[A, B] = \phi^{-1}(I_1I_2)$ may be elliptic, parabolic, or loxodromic. The representation is discrete and faithful when $[A, B]$ is parabolic or loxodromic. Passing to the index 3 subgroup, this is the same as saying $I_1I_2I_3 = (I_1J)^3$ is parabolic and I_1I_2, I_2I_3 and I_3I_1 are all parabolic or loxodromic. The statement is

Theorem 5.2 ([2])**.** *Suppose that* $\rho : \Delta(\infty,\infty,\infty) \longrightarrow \mathrm{PU}(2,1)$ *is a representation so that* $I_j = \rho(\iota_j)$ *fixes a complex line and* $I_1I_2I_3 = \rho(\iota_1\iota_2\iota_3)$ *is parabolic. Suppose that there exists a symmetry map* J *of order* 3 *so that* $I_2 = JI_1J^{-1}$ *and* $I_3 = J^{-1}I_1J$. *Then* ρ *is discrete and faithful if and only if* $I_1I_2 = \rho(\iota_1\iota_2)$ *(and so also* I_2I_3 *and* I_3I_1*) is loxodromic or parabolic.*

6. THE HOROTUBE SURGERY THEOREM

We will now describe the main result of Schwartz's book, the horotube surgery theorem. We give a precise statement in Theorem 6.1 below. Roughly speaking, the idea behind this theorem is that one begins with a cusped 3-manifold or orbifold with a spherical CR structure and then by performing certain Dehn surgeries, one constructs new manifolds or orbifolds which have spherical CR structures.

To be more precise, the class of groups to which the horotube surgery theorem applies are what Schwartz calls horotube representations of an abstract group Γ. We will now discuss the properties of a horotube representation. Consider a representation $\rho_0 : \Gamma \longrightarrow \mathrm{PU}(2,1)$. Suppose that $P \in \rho_0(\Gamma)$ is a parabolic map with fixed point $p \in \partial\mathbf{H}^2_{\mathbb{C}}$. A *horotube* is a P-invariant open set T of $\partial\mathbf{H}^2_{\mathbb{C}} - \{p\}$ so that $T/\langle P\rangle$ has a compact complement in $\big(\partial\mathbf{H}^2_{\mathbb{C}} - \{p\}\big)/\langle P\rangle$. Schwartz calls the quotient $T/\langle P\rangle$ a *horocusp*. Suppose that $\rho_0(\Gamma)$ is discrete, and write Λ for its limit set and Ω for its domain of discontinuity in $\partial\mathbf{H}^2_{\mathbb{C}}$. Then Ω is *porous* if there exists $\epsilon_0 > 0$ so that $A(\Omega)$ contains a ball of spherical diameter ϵ_0 for all $A \in \mathrm{PU}(2,1)$. This condition should be equivalent to Γ being geometrically finite with no maximal rank cusps (see page 28 of [13]). A discrete representation $\rho_0 : \Gamma \longrightarrow \mathrm{PU}(2,1)$ is a *horotube representation* if every elliptic element of $\rho_0(\Gamma)$ has a unique fixed point in $\mathbf{H}^2_{\mathbb{C}}$, the domain of discontinuity Ω is porous, and its quotient $\Omega/\rho_0(\Gamma)$ is the union of a compact set together with a finite collection of disjoint horocusps. In particular, if ρ_0 is a horotube representation, then every parabolic subgroup of $\rho_0(\Gamma)$ is cyclic.

The horotube surgery theorem concerns families of representations of Γ that converge to ρ_0. Suppose that ρ_0 is a horotube representation of Γ. An infinite cyclic subgroup Υ of Γ is *peripheral* if $\rho_0(\Upsilon)$ is a parabolic subgroup. Such groups are in one-to-one correspondence with the horocusps. Schwartz says that a sequence of representations $\rho_n : \Gamma \longrightarrow \mathrm{PU}(2,1)$ for $n \in \mathbb{N}$ converges nicely to ρ_0 if for all $\gamma \in \Gamma$ and all peripheral subgroups $\Upsilon < \Gamma$

- $\rho_n(\gamma) \longrightarrow \rho_0(\gamma)$ geometrically for each $\gamma \in \Gamma$;
- $\rho_n(\Upsilon) \longrightarrow \rho_0(\Upsilon)$ setwise with respect to the Hausdorff topology;
- if $\rho_n(\Upsilon)$ is finite, then each of its elements has a unique fixed point in $\mathbf{H}^2_{\mathbb{C}}$.

Theorem 6.1 (Horotube surgery, Theorem 1.2 of [13])**.** *Suppose that we are given a horotube representation* $\rho_0 : \Gamma \longrightarrow G_0 < \mathrm{PU}(2,1)$*. Let* $\rho_n : \Gamma \longrightarrow G_n < \mathrm{PU}(2,1)$ *be a sequence of representations that converges nicely to* ρ_0*. Then there exists* N *so that if* $n \geq N$*, the group* $G_n = \rho_n(\Gamma)$ *is discrete and* Ω_n/G_n *is obtained from* Ω_0/G_0 *by performing a Dehn filling on each horocusp of* Ω_0/G_0 *corresponding to a peripheral subgroup* Υ *for which* $H_n = \rho_n(\Upsilon)$ *is not parabolic. If at least one cusp is not filled, then* ρ_n *is a horotube representation of* $\Gamma/\ker(\rho_n)$*.*

Furthermore Schwartz gives precise details about which Dehn surgeries arise in terms of $\rho_0(\Upsilon)$ and $\rho_n(\Upsilon)$.

7. Application of the HST to triangle groups

We now discuss how the horotube surgery theorem may be applied to triangle groups. The starting point is the golden triangle group. Schwartz proves that this is a horotube representation with four (conjugacy classes of) peripheral subgroups, namely $\Upsilon_{12} = \langle \iota_1\iota_2 \rangle$, $\Upsilon_{23} = \langle \iota_2\iota_3 \rangle$, $\Upsilon_{31} = \langle \iota_3\iota_1 \rangle$, and $\Upsilon_{123} = \langle \iota_1\iota_2\iota_3 \rangle$. Suppose that $\rho_n(\Delta)$ is a sequence of representations of $\Delta = \Delta(\infty,\infty,\infty)$ converging nicely to the golden triangle group. Then there are several possible scenarios depending on whether the generator of each of these subgroups is loxodromic, elliptic, or parabolic.

For example, suppose that the three peripheral subgroups Υ_{jk} are all parabolic and Υ_{123} is loxodromic. Such representations are covered by Theorem 4.2, which indicates that they are all horotube representations. Likewise, suppose that Υ_{jk} are all loxodromic and Υ_{123} is parabolic. If, in addition, there is a symmetry map J that cyclically conjugates $\rho(\Upsilon_{12})$, $\rho(\Upsilon_{23})$, and $\rho(\Upsilon_{31})$, then such representations are covered by Theorem 5.2, which indicates that they are all horotube representations.

The more interesting case arises when at least one of the peripheral subgroups is elliptic. In the symmetric case, there are only finitely many discrete representations where all the peripheral subgroups are elliptic, and so we cannot use the horotube surgery theorem in this case.

Theorem 7.1 ([7])**.** *There are only finitely many conjugacy classes of symmetric, discrete representations* $\rho : \Delta(p,p,p) \longrightarrow \mathrm{PU}(2,1)$ *for which* $\rho(\iota_j) = I_j$ *fixes a complex line and for which* $I_1I_2 = \rho(\iota_1\iota_2)$ *and* $I_1I_2I_3 = \rho(\iota_1\iota_2\iota_3)$ *are both elliptic.*

Of course, we could ask about asymmetric groups where all the peripheral subgroups of the golden triangle group are elliptic.

Therefore, it is natural to ask about groups for which one family of peripheral triangle groups is elliptic and the other loxodromic. This is one of the applications of the horotube surgery theorem given by Schwartz. Consider one of the groups from Theorem 4.2 for which $I_1I_2I_3 = \rho(\iota_1\iota_2\iota_3)$ is loxodromic and $I_jI_k = \rho(\iota_j\iota_k)$ is parabolic for each pair $j \neq k$ in $\{1, 2, 3\}$. This group is the limit of a sequence of representations with $I_1I_2I_3$ loxodromic and at least one of the I_jI_k elliptic, the orders tending to infinity. Now suppose that ρ is sufficiently far along this sequence.

By applying the horotube surgery theorem, Schwartz is able to prove the following result.

Theorem 7.2 (Theorem 1.10 of [13])**.** *Suppose that* $\rho : \Delta(p,q,r) \longrightarrow \mathrm{PU}(2,1)$ *is a representation so that* $I_j = \rho(\iota_j)$ *fixes a complex line and* $I_jI_k = \rho(\iota_j\iota_k)$ *has the same order as* $\iota_j\iota_k$ *(*p*,* q*, and* r *may also be* ∞*). Suppose also that* $I_1I_2I_3 = \rho(\iota_1\iota_2\iota_3)$ *is loxodromic. Then for* $\min\{p, q, r\}$ *sufficiently large,* ρ *is a horotube representation and hence is discrete.*

We can give a further application of the horotube surgery theorem by swapping the roles of I_jI_k and $I_1I_2I_3$ in the previous theorem. Namely, consider one of the symmetric representations in Theorem 5.2 where I_1I_2, I_2I_3, and I_3I_1 are each loxodromic and $I_1I_2I_3$ is parabolic. This group is the limit of a sequence of groups for which I_1I_2, I_2I_3, and I_3I_1 are loxodromic and $I_1I_2I_3$ is regular elliptic. Since our original group is a horotube representation, by taking ρ sufficiently far along this sequence, we can apply the horotube surgery theorem. This leads to the following result:

Theorem 7.3. *Suppose that* $\rho : \Delta(\infty,\infty,\infty) \longrightarrow \mathrm{PU}(2,1)$ *is a symmetric representation for which* $I_j = \rho(\iota_j)$ *fixes a complex line and* $I_jI_k = \rho(\iota_j\iota_k)$ *is loxodromic. If* $\rho(I_1I_2I_3)$ *is regular elliptic of sufficiently high order, then* ρ *is a horotube representation and hence is discrete.*

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