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Hyperbolic geometry from a local viewpoint, by Linda Keen and Nikola Lakic, London Mathematical Society Student Texts, 68, London Mathematical Society, Cambridge University Press, Cambridge, 2007, x+271 pp., ISBN 13: 978-0-521-68224-4, US $47.00 (paperback), ISBN 13: 978-0-521-86360-5, US $121.00 (hardback)

In addition to being packed with fundamental material important for every beginner in complex analysis, in expeditious and intuitive terms this little book transports the reader through a range of interesting topics in one-dimensional hyperbolic geometry, discrete subgroups, holomorphic dynamics and iterated function systems. In chapter after chapter, one quickly arrives at open problems and areas for research.

The first half of the book carries the reader through the essential elements of Riemann surface theory, starting with geometry in the Euclidean plane and the Riemann sphere and going on to hyperbolic geometry in the hyperbolic plane. Topics include basic properties of holomorphic and univalent functions, Schwarz's lemma, covering spaces, universal covering spaces, fundamental groups, discontinuous groups, Fuchsian groups.

There is a very nice exposition of the Poincaré polygon theorem, which provides a sufficient condition for a subgroup of $PSL(2,\mathbb{R})$ generated by side-pairings of a hyperbolic polygon to form a discrete group. There is also a concise and elementary exposition of the collar lemma. The collar lemma is a fundamental lemma for the analysis of hyperbolic surfaces. It provides a collar of definite thickness containing any closed geodesic on a hyperbolic surface, and as the geodesic shortens the collar gets thicker. One of its consequences is that there is a fixed lower bound on the lengths of intersecting closed geodesics. The lower bound is universal for all surfaces and all geodesics, and in this respect the lemma resembles the Heisenberg uncertainty principle. In Bers [4, page 443-449] one can find a history of the lemma and references for it.

In the second half of the book, starting with the chapters on Kobayashi and Carathéodory metrics for hyperbolic plane domains, there are great numbers of new theorems, many of which are only recently proved and some of which appear for the first time in this publication. In this part the approach is more categorical. The basic objects of study are canonical procedures for defining conformal metrics on Riemann surfaces. The chief constraint is that such procedures are required to give metrics that satisfy the conclusion of Schwarz's lemma. If X is a Riemann surface and we denote by σ_X the infinitesimal form of the canonically associated metric, we require that for any holomorphic map f mapping X into another Riemann surface Y we have the inequality

$$\sigma_Y(f(z))|f'(z)| \leq \sigma_X(z). \tag{1}$$

If σ satisfies this condition, we say it has the Schwarz lemma property.

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Whenever we have an infinitesimal form, we obtain a distance between two points p and q in X by putting

$$\sigma_X(p,q) = \inf\left\{\int_\gamma \sigma_X(z)|dz|\right\},$$

where the infimum is taken over all paths γ joining p to q. Then $\sigma_X(p,q)$ is symmetric and satisfies the triangle inequality. In some cases, $\sigma_X(p,q)$ may only be a pseudometric. That is, possibly $\sigma_X(p,q)$ assigns the value 0 even when $p \neq q$, and in other cases $\sigma_X(p,q)$ might be equal to ∞. Even so, if the Schwarz lemma property is satisfied for the infinitesimal metric, then its corresponding version is also satisfied for the metric; that is,

$$\sigma_Y(f(p), f(q)) \leq \sigma_X(p,q),$$

whenever $f : X \to Y$ is holomorphic.

The first example is the Poincaré infinitesimal form, ρ_X. In the unit disc, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, this form is given by

$$\rho_\Delta(z)|dz| = \frac{|dz|}{1-|z|^2}.$$

It has the property that any holomorphic automorphism B of Δ is an isometry; that is, $\rho_\Delta(B(z))|B'(z)| = \rho_\Delta(z)$. The metric associated to this infinitesimal form is

$$\rho_\Delta(p,q) = \frac{1}{2}\log\frac{1+\left|\frac{p-q}{1-\overline{q}p}\right|}{1-\left|\frac{p-q}{1-\overline{q}p}\right|}.$$

If $\pi : \Delta \to X$ is a universal covering of a Riemann surface X, by definition $\rho_X(p) = \frac{\rho_\Delta(z)}{|\pi'(z)|}$ where $\pi(z) = p$. The meaning of $\pi'(z)$ depends on a choice of local parameter at p. If w is such a parameter, then

$$\pi'(z) = \lim_{t\to z}\frac{w\circ\pi(t) - w\circ\pi(z)}{t-z}.$$

With this understanding, it is easy to check that $\rho_X(w)|dw|$ is an invariant form. Universal covering maps have the lifting property; that is, any continuous map $f : X \to Y$ has a continuous lift $\tilde{f} : \Delta \to \Delta$ such that $\pi_Y \circ \tilde{f} = f \circ \pi_X$, and $\tilde{f}$ is holomorphic if f is. The Schwarz-Pick lemma says that for holomorphic self-map g of the unit disc,

$$\rho_\Delta(g(z))|g'(z)| \leq \rho_\Delta(z). \tag{2}$$

Applying this to $g = \tilde{f}$, one obtains

$$\rho_Y(f(z))|f'(z)| \leq \rho_X(z). \tag{3}$$

We conclude that the Poincaré infinitesimal form ρ_X is assigned to any Riemann surface X whose universal covering is conformal to the unit disc, and this assignment has the Schwarz lemma property.

A second example is the Kobayashi infinitesimal form $\kappa_X(z)|dz|$. By definition

$$\kappa_X(w) = \inf\left\{\frac{\rho_\Delta(z)}{|f'(z)|}\right\}, \tag{4}$$

where the infimum is taken over all holomorphic functions f mapping the unit disc Δ into X and mapping z to w. From (3), if the surface has a universal covering conformal to the unit disc, then $\rho_X(w) \leq \frac{\rho_\Delta(z)}{|f'(z)|}$, so from (4) $\rho_X(w)|dw| \leq \kappa_X(w)|dw|$ and κ_X is positive. From the uniformization theorem, this is the only way (4) can be positive, and in that case $\rho_X(w)|dw| = \kappa_X(w)|dw|$. When it is not positive, $\kappa_X(p) = 0$ for every point $p \in X$ and X is a Riemann surface with universal covering conformal either to the complex plane $\mathbb{C}$ or to the extended complex plane $\overline{\mathbb{C}}$. Surfaces for which $\kappa_X(p) > 0$ are called hyperbolic. So there is nothing really new about this example except that the extremal problem for finding κ_X also applies to non-hyperbolic surfaces where it gives $\kappa_X(p, q)$ identically equal to zero. One slightly new observation is that the Schwarz lemma property still holds.

A third example is the Carathéodory metric, c_X. Once again we take the Poincaré metric and the Schwarz inequality as the starting points, but instead of using the family of holomorphic maps from Δ to X, we use maps from X to Δ and a dual extremal problem. By definition

$$c_X(p) = \sup \{\rho_\Delta(g(p))|g'(p)|\}, \tag{5}$$

where the supremum is over all holomorphic maps g from X to Δ. By parallel arguments we obtain $c_X \leq \rho_X$, and in general we have

$$c_X \leq \rho_X \leq \kappa_X$$

for every Riemann surface X. The metric c_X turns out to be a really different example. When X is an annulus, one can easily show that $c_X < \rho_X$, although even in this case explicit calculation of c_X seems to be difficult. The topology on X induced by $c_X(p, q)$ coincides with the topology induced by $\rho_X(p, q)$ and c_X has the Schwarz lemma property.

One obtains many more examples by using extremal problems similar to those of Kobayashi and Carathéodory if one replaces the reference surface Δ by some other hyperbolic Riemann surface Ω. In particular, Keen and Lakic define relative Kobayashi and Carathéodory metrics the following way:

$$\kappa_X^\Omega(w) = \inf \left\{ \frac{\rho_\Omega(z)}{|f'(z)|} : f \textit{ holomorphic from } \Omega \textit{ into } X \textit{ with } f(z) = w \right\},$$
$$c_X^\Omega(z) = \sup \left\{ \rho_X(f(z))|f'(z)| : f \textit{ holomorphic from } \Omega \textit{ into } X \right\}.$$

κ_X^Ω and c_X^Ω are canonical infinitesimal forms, and they satisfy the Schwarz lemma property in the variables X and Ω, respectively.

When X is a subset of Ω, these definitions become useful for the study of iterations of holomorphic maps from Ω into X. Depending on geometric properties of X and Ω, it is possible that the inclusion map from X into Ω is a strict contraction, by which we mean there might be a constant $l(X, \Omega)$ such that

$$\sup_{z \in X} \frac{\rho_\Omega(z)}{\rho_X(z)} = l(X, \Omega) < 1.$$

If this is so, any holomorphic map $f : \Omega \to X$ is a strict contraction both for the metric ρ_Ω and ρ_X.

Definition. X is called a Lipschitz subdomain of Ω if $l(X, \Omega) < 1$.

Definition. Let $R(X,\Omega)$ be the supremum of the radii of hyperbolic discs with respect to the metric ρ_Ω completely contained in X. X is called a (hyperbolic) Bloch domain in Ω if $R(X,\Omega) < \infty$.

In [3] Beardon, Carne, Minda and Ng prove the following theorem.

Theorem 1. *X is a Bloch subdomain of Ω if, and only if, it is a Lipschitz subdomain of Ω; that is $l(X,\Omega) < 1$ if and only if $R(X,\Omega) < \infty$.*

Since the notion of a subdomain being Bloch or Lipschitz is defined in terms of the Poincaré metric, one can make parallel definitions in terms of either the Kobayashi or Carathéodory metrics.

Definition. X is called a Kobayashi-Lipschitz or Carathéodory-Lipschitz subdomain of Ω if

$$\sup_{z\in X} \frac{\rho_\Omega(z)}{\kappa_X^\Omega(z)} < 1 \text{ or } \sup_{z\in X} \frac{c_X^\Omega(z)}{\rho_X(z)} < 1, \text{ respectively.}$$

Definition. Let $cR(X,\Omega)$ and $kR(X,\Omega)$ be the supremum of the radii of discs measured with respect to the metric c_X^Ω or κ_Ω^Ω, respectively, completely contained in X. Then X is called a Carathéodory-Bloch or Kobayashi-Bloch domain in Ω if $cR(X,\Omega) < \infty$ or $kR(X,\Omega) < \infty$, respectively.

Keen and Lakic prove many theorems straightening out the relationships between these concepts. For example, they prove Lipschitz, Bloch and Kobayashi-Bloch are equivalent concepts for $X \subset \Omega$, Lipschitz implies Carathéodory-Lipschitz, Lipschitz implies Kobayashi-Lipschitz and Kobayashi-Bloch implies Carathéodory-Bloch. By providing counterexamples they also show in many cases that these conditions are not necessary and sufficient.

If X is a subset of Ω, these metrics become useful for studying the convergence properties of a composition of a randomly selected sequence of holomorphic functions f_n mapping Ω into X. Let

$$G_n = f_n \circ \cdots \circ f_1 \text{ and } F_n = f_1 \circ \cdots \circ f_n.$$

F_n and G_n are called the backward and forward iterations, respectively. Forward iterations are relatively simple to understand and Keen and Lakic show that all accumulation points of any forward iteration of maps from a hyperbolic plane domain Ω into a subdomain X are constant functions if and only if $\Omega \neq X$. Moreover, any closed subset of $\overline{X}$ can be realized as the set of forward accumulation points of some iterated function system. These results are new and can be viewed as broad generalizations of the Denjoy-Wolff theorem which states that for any holomorphic function that maps the unit disc into itself and is not a Möbus transformation, the iterates converge to some value in the closed unit disc.

The situation for backward iterations is more interesting. The question is to find the most general possible condition on $X \subset \Omega$ that guarantees that any backward iteration system has only constant accumulation points. A subdomain with this property is called degenerate. Here, Keen and Lakic produce a beautiful theorem by inventing a condition that combines the relationship between the Carathéodory and Kobayashi metrics relative to a given domain Ω. First of all, notice that

$$c_X^\Omega \leq \rho_\Omega \leq \rho_X \leq \kappa_X^\Omega, \tag{6}$$

and the condition that X is Kobayashi-Lipschitz in Ω is that

$$\sup_{z \in X} \frac{\rho_\Omega(z)}{\kappa_X^\Omega(z)} < 1, \tag{7}$$

and the condition that X is Carathéodory-Lipschitz in Ω is that

$$\sup_{z \in X} \frac{c_X^\Omega(z)}{\rho_X(z)} < 1. \tag{8}$$

Keen and Lakic put

$$ck = \sup_{z \in X} \frac{c_X^\Omega(z)}{\kappa_X^\Omega(z)},$$

and so by using (6) one sees that the condition that $ck < 1$ is weaker than either of the previous conditions (7) or (8). They call a subdomain X of Ω for which $ck < 1$ a ck-Lipschitz domain and show that such domains are denerate. The idea of a ck-Lipschitz domain is that maybe on one part of X the Carathéodory condition is satified and on another part the Kobayashi condition is satisfied, but neither of these conditions is separately satisfied on all of X. The ck-condition allows for one of the Carathéodory or Kobayashi conditions to take over wherever the other one fails.

The proof that such domains are degenerate is straightforward. From the Schwarz lemma properties

$$k_X^\Omega(f(z))|f'(z)| \leq k_\Omega^\Omega(z)$$

and

$$c_X^X(f(z))|f'(z)| \leq c_X^\Omega(z),$$

and since $k_\Omega^\Omega(z) = \rho_\Omega(z)$ and $c_X^X(z) = \rho_X(z)$, one has the inequality

$$|f'(z)|^2 \leq \frac{\rho_\Omega(z)}{k_X^\Omega(z)} \frac{c_X^\Omega(z)}{\rho_X(f(z))}.$$

By applying this inequality to the derivative of a long composition and using the ck-constant, one shows that the derivative of any limit must be constant.

This theory suggests trying to develop a similar approach to the Kobayashi and Carathéodory metrics for holomorphic maps of Teichmüller space into subdomains. In this setting, it is not known if the Carathéodory and Kobayashi metrics coincide, although there is an interesting positive result in this direction due to Kra, [11]. Theorems that provide sufficient conditions for existence of fixed points of holomorphic self-maps of Teichmüller space into itself can be useful.

After the chapters concerning relative Kobayashi and Carathéodory metrics for plane domains and iterated function systems, there are two additional chapters: one that gives explicit lower bounds for Poincaré metrics of plane domains and a second on the theory of uniformly perfect domains. Finally, there is a nice appendix giving a brief exposition of elliptic functions.

This book is a basic reference for background and problems in one-dimensional dynamics and complex analysis along with many other such books. We have included references for some of these in the bibliography, [1], [2], [6], [7], [10], [12], [13], [14], [15], [16], [17].

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