

## SELECTED MATHEMATICAL REVIEWS

related to the paper in this section by

H. WU

MR0006075 (3,253) 52.0X

**Chern, Shiing-shen**

**On integral geometry in Klein spaces.**

*Ann. of Math. (2)* **43** (1942), 178–189.

This paper constitutes a praiseworthy attempt to bring methods and results of the so-called “integral geometry” where they really belong, that is, within the framework of E. Cartan’s theory of homogeneous spaces. Those (called Klein spaces by the author) are the spaces in which a transitive Lie group  $G$  of automorphisms has been defined, and which can therefore be represented as spaces of cosets  $G/g$  in  $G$  with respect to a subgroup  $g$ . As the author has realized, integral geometry is the study of a certain type of relations between integral invariants in two homogeneous spaces  $N = G/g$ ,  $M = G/h$  defined by the same group  $G$  and different subgroups  $g$ ,  $h$ , and is by no means confined to the group of motions in a Euclidean space (or in one of constant curvature). The best-known case, and the one which may best serve as illustration, is of course that in which  $G$  is the group of motions in Euclidean space  $E^n$ , where  $n = 2$  or  $3$ , and  $g$ ,  $h$  are the subgroups which respectively leave invariant a given straight line and a given point;  $G/h$  is then  $E^n$  itself (viewed as a point-space), and  $G/g$  the space of straight lines in  $E^n$ .

§ 1 sketches in brief outline Cartan’s method of invariant differential forms as applied to homogeneous spaces; it gives, in terms of the constants of structure, a necessary and sufficient condition for the existence of an invariant measure, expressible by the integral of an invariant differential form, in a homogeneous space [cf. the treatment of the same problem for arbitrary locally compact groups in the recent monograph by the reviewer, “L’intégration dans les groupes topologiques,” *Actual. Sci. Ind.*, no. 869, Hermann et Cie., Paris, 1941; MR0005741 (3,198b)]. The condition, as given, is purely local; the further considerations which are necessary if orientation is taken into account ( $g$  need not be connected, nor need  $G/g$  be orientable) are omitted.

§ 2 proposes a general definition of “incidence” between an element  $a$  in  $M = G/h$  and an element  $b$  in  $N = G/g$ ;  $a$ ,  $b$  are called “incident” if the cosets  $x \cdot h$ ,  $y \cdot g$  which define them have a point  $z$  in common, in which case they of course have in common the whole coset  $z \cdot \gamma$  with respect to  $\gamma = h \cap g$ . This amounts to saying that there is a transformation  $u = z^{-1}$  in the group which brings  $a$ ,  $b$  to a given canonical position  $u(a) = a_0$ ,  $u(b) = b_0$  (corresponding to the particular cosets  $h$ ,  $g$ ). This clearly includes the usual definition of “incidence” in all relevant particular cases, but is much more general. In the case of points and straight lines in  $E^n$ , for instance, it will mean that a point and straight line are called “incident” in the sense of the author if they are at a given distance  $\delta$  from each other,  $\delta$  being the distance between  $a_0$  and  $b_0$ ; this is the same as the usual “incidence” if  $\delta = 0$ ; still calling it “incidence” for  $\delta > 0$  may be slightly misleading, but only serves to widen the scope of the author’s results.

§ 3 is an attempt to prove the analogue of Crofton's and Cauchy's formulae in the general setting previously outlined; the generalized Crofton formula alone is discussed in full detail. Crofton's classical theorem asserts that the measure of the set of straight lines in  $E^2$  which are incident with a given curve is, up to a universal constant factor, the length of the curve; the same is true in  $E^3$ , curves and their lengths being replaced by surfaces and their areas. The author now considers the two homogeneous spaces  $M = G/h$ ,  $N = G/g$ , of dimensions  $m$ ,  $n$ , respectively; let  $V_p$  be a portion of a  $p$ -dimensional variety in  $M$ , the dimension  $p$  being such that the set  $V_n$  of elements of  $N$  which are incident with some elements of  $V_p$  is of dimension  $n$ . The main part of the author's argument then consists in finding a suitable expression of the differential form which defines the invariant measure in  $N$ , when that form is to be calculated over  $V_n$ . To that effect, a system of representatives  $W_p$  is chosen in  $G$  for the cosets which correspond to the elements of  $V_p$ ; the calculation then proceeds correctly, although notations are somewhat confusing, and the advantages of Cartan's methods are partly lost by the unnecessary introduction of local coordinates and the implicit use of systems of representatives; the assertions [p. 185] that the forms  $\bar{\omega}^i$  are "Pfaffian forms in  $\lambda^v$  only," and that  $a_k^i$  "are functions of  $\lambda^v$ ," would be trivial if systems of representatives had been chosen in  $h$  for the cosets of  $\gamma$ , but are untrue otherwise. Most of the calculation is valid for an arbitrary choice of the system of representatives  $W_p$  for  $V_p$  in  $G$ ; the author, however, assumes from the beginning that this choice has been made in a definite manner "by the method of moving frames of Cartan," and uses this in trying to define the invariant  $p$ -dimensional measure (or "area") of  $V_p$  in  $M$ ; unfortunately, the treatment is exceedingly sketchy at this point, and the phrase "it is in general possible" [p. 182.] hardly helps to dissipate the obscurities of the proposed definition. There is even more vagueness in the carrying out of the integration of the calculated differential form; the author only says that he does this "by first holding an element of  $V_p$  fixed"; doing so without any precaution, however, would give 0 even in such a simple case as that of the classical Crofton formula; in the latter case, what is done is to integrate, not the form itself, but its absolute value; and it is open to question whether the same device should be adopted in the general case, or whether (as the author seems to suggest in the case of infinite integrals) integration should be carried out over some part only of the set  $V_n$ . It is apparent, from the work of G. de Rham on integral invariants [Jber. Deutsch. Math. Verein. **49**, 156–161 (1939); MR0000721 (1,119e)] that the invariant  $p$ -dimensional measure in a homogeneous space  $M$  is not always uniquely defined, and therefore deeper investigations probably would be required in order to settle those questions [cf. also W. Maak's work on Crofton's formula, Math. Ann. **116**, 574–597 (1939)]. It would seem, therefore, that the author has not conclusively proved any general theorem on the subject; however, he has given a method which is probably adequate to settle any particular case where we already have some information about the application of the moving frame method; and it should again be emphasized that the main calculation is correct and should be found useful in any further work on the subject. Similar remarks apply to the author's attempt at a generalization of Cauchy's formula, which is given only in brief outline, and depends upon the same calculation as above indicated.

**MR0192436 (33 #661)** 53.52

**Chern, S. S.**

**The geometry of  $G$ -structures.**

*Bull. Amer. Math. Soc.* **72** (1966), 167–219.

This paper is a revised and expanded version of the American Mathematical Society Colloquium Lectures of 1960. It constitutes a remarkably comprehensive and useful survey of modern work on riemannian and complex structures in differential geometry. The unifying notion of a  $G$ -structure on an  $n$ -manifold  $M$  is emphasised, by which the author means a reduction of the group  $GL(n, R)$  to the subgroup  $G$ . He then considers the basic general problems of existence of particular  $G$ -structures, the local and global properties of those which exist, and the mappings between structures. The author remarks that he “will emphasize simple and concrete problems at the expense of generality”. Since the paper is itself a review, one can add little except to praise its utility, interest and clarity. It contains a large number of examples, and descriptions of theorems; several problems are stressed; and there is a list of 141 references. There are 13 sections, listed, with brief indication of contents, as follows: Introduction; Riemannian structure (curvature, Gauss-Bonnet theorem, pinched manifolds); Connexions (and holonomy groups);  $G$ -structures (Cartan’s problem of equivalence); Harmonic forms (holonomic  $G$ -structures); Leaved structures (here regarded as a reduction of the structural group with differential conditions, with examples from topological dynamics); Complex structure (many examples of complex manifolds); Sheaves (Cousin’s problem, de Rham’s Theorem, Dolbeault cohomology, Stein manifolds, classification of bundles by  $H^1(M, S)$ ); Characteristic classes (conditions for reduction of structural group, operations on vector bundles, curvature, divisibility); Theorems of Riemann-Roch, Hirzebruch, Grothendieck, and Atiyah-Singer; Holomorphic mappings of complex-analytic manifolds (theorems of Chow, Kodaira, Plücker, Picard, Ahlfors, Levine); Isometric mappings of Riemannian manifolds (Nash’s theorem, tight immersions and embeddings, minimal submanifolds); General theory of  $G$ -structures (local flatness, Pythagorean nature of metric).

It should be clear then, that here is the outline of a large book, which one hopes the author will someday complete.

From MathSciNet, December 2008

*H. B. Griffiths*

**MR0353327 (50 #5811)** 57D20

**Chern, Shiing Shen; Simons, James**

**Characteristic forms and geometric invariants.**

*Ann. of Math. (2)* **99** (1974), 48–69.

This paper is a detailed version of a previous one [Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 791–794; MR0279732 (43 #5453)]. In the intervening period the activity on this topic of secondary characteristic classes has been great, so it seems proper to examine here some of the details and indicate a little of where and how it has been pursued.

Let  $E \rightarrow M$  be a principal bundle with group  $G$ . For each invariant polynomial  $P \in I^k(G)$ , the Weil homomorphism gives a characteristic class, determined by the invariant  $2k$ -form  $P(\Omega^k)$  on  $E$ , where  $\Omega$  is the curvature form of a connection

$\theta$  on  $E$ . The authors define a  $(2k - 1)$ -form  $TP(\theta)$  on  $E$  having  $P(\Omega^k)$  as its exterior derivative. On each fiber  $E_m$  the restriction of  $TP(\theta)$  is a closed form that is independent of the choice of  $\theta$  and represents the transgression class in  $H^{2k-1}(G; \mathbf{R})$  which corresponds to  $P$ . An explicit formula is given for  $TP(\theta)$ , but perhaps it is more important that it is determined up to an exact form by the following functorial properties in the category of  $G$ -bundles with a connection: (1)  $dTP(\theta) = P(\Omega^k)$  and (2)  $P \rightarrow TP(\theta)$  is natural with respect to connection-preserving  $G$ -bundle maps. The authors' object is to give geometrical significance to the forms  $TP(\theta)$ , especially in the case where  $P(\Omega^k) = 0$  so that  $TP(\theta)$  is closed and represents a cohomology class on  $E$ .

When  $P$  is integral, then the information carried by  $TP(\theta)$  can be pushed forward to  $M$  in the form of a cochain  $u \in C^{2k-1}(M; \mathbf{R}/\mathbf{Z})$  such that, up to a coboundary,  $\pi^*u$  is the reduction of  $TP(\theta) \bmod \mathbf{Z}$ . The algebra of such cochains (differential characters) has been studied further by J. Cheeger and the second author ("Differential characters and geometric invariants", to appear) and Cheeger [Symposia Mathematica, Vol. XI (Convegno di Geometria, INDAM, Roma, Maggio, 1972), pp. 441–445, Academic Press, London, 1973]. If  $2k - 1 \geq \dim M$ , so that  $P(\Omega^k) = 0$ , then  $u$  is a cocycle. Sometimes  $u$  carries information independent of the choice of  $\theta$ ; this is true of the cohomology class of  $u$  when  $2k - 1 > \dim M$ , and when  $\dim M = 2k - 1$ , there are general conditions under which the reduction of  $u \bmod Q$  is independent of  $\theta$ .

When  $E$  is the bundle of bases, then the significant invariant polynomials are those giving the Pontrjagin classes. Then the authors prove that if  $\theta$  and  $\theta'$  are the Riemannian connections of conformally related metrics on  $M$ ,  $TP(\theta)$  and  $TP(\theta')$  differ by an exact form and  $P(\Omega^k) = P(\Omega'^k)$ . Hence when  $P(\Omega^k) = 0$ ,  $TP(\theta)$  represents a cohomology class in  $H^{2k-1}(E; \mathbf{R})$  which is a conformal invariant.

As an application a necessary condition for conformal immersion of a Riemannian manifold in  $\mathbf{R}^{n+k}$  is obtained, where  $n = \dim M$ . If there is such an immersion, the normal Pontrjagin forms  ${}^\perp P_i(\Omega^{2i}) = 0$  for  $i > [k/2]$ , and  $\frac{1}{2}T^\perp P_i(\theta)$  represents an integral cohomology class on  $E$ . A calculation shows that for a bi-invariant metric on  $\text{SO}(3)$ ,  $\frac{1}{2}TP_1(\theta) = -\frac{1}{2}T^\perp P_1(\theta)$  is not integral, and consequently  $\text{SO}(3)$  does not admit a conformal immersion into  $R^4$ . For an extension of this result to the classical compact groups see the paper of J. L. Heitsch and H. B. Lawson, Jr. reviewed below [#5812]. An oriented compact 3-manifold is parallelizable, so that  $\frac{1}{2}TP_1(\theta)$  can be pulled down to  $M$  via a section of  $E$ ; the integral of this pulled-down form is an  $\mathbf{R}/\mathbf{Z}$ -invariant  $\Phi(g)$  of the conformal class of the metric  $g$ . The authors show that as a function of metrics,  $\Phi$  has  $g$  as a critical point if and only if  $g$  is conformally flat.

The effective use of this theory seems to exploit two hypotheses: an integrality assumption on  $P$  and the vanishing of  $P(\Omega^k)$ . In this paper, this vanishing is obtained under a dimension hypothesis. However, it is possible to obtain more extensive applications when one has Bott-type vanishing theorems, that is, in the case of foliated bundles. The systematic study of such vanishing theorems and the secondary characteristic classes they lead to is being pursued by F. W. Kamber and P. Tondeur [see, e.g., Manuscripta Math. **11** (1974), 51–89; MR0334237 (48 #12556)].

**MR0494957 (80b:53008)** 53A60; 14C21, 14H99, 53B25

**Chern, S. S.; Griffiths, Phillip**

**Abel's theorem and webs.**

*Jahresber. Deutsch. Math.-Verein.* **80** (1978), no. 1-2, 13–110.

This paper is in the tradition of Blaschke and his school. The subject is the local invariants of a finite family of foliations; the methods are those of algebraic, projective, and differential geometries.

A  $d$ -web on an open set  $U$  in  $\mathbf{R}^n$  is a set of  $d$  foliations of  $U$ , each foliation of codimension one. The leaves of the foliations are always assumed to be in general position. Two  $d$ -webs are equivalent if one can be mapped to the other by a smooth diffeomorphism of the domains of definition. For  $d \leq n$ , all  $d$ -webs are locally equivalent. The most interesting case is when  $d > 2n \geq 6$  and this is assumed in the following. A  $d$ -web is linear if the leaves are all hyperplanes and is linearizable if it is equivalent to a linear  $d$ -web. Note that two linear  $d$ -webs are generally not equivalent.

This study of  $d$ -webs is motivated by the following example. (In the manner of the paper we intermix real and complex structures. Also all statements should be interpreted locally.) Consider an algebraic curve  $C$  of degree  $d$  in  $\mathbf{P}^n$ . Let  $\xi \in \mathbf{P}^{n*}$  be a generic hyperplane and  $p_1(\xi), \dots, p_d(\xi)$  the points of intersection of  $\xi$  with  $C$ . The hyperplanes in  $\mathbf{P}^{n*}$  dual to each  $p_i(\xi)$  define a linear  $d$ -web on an open set  $U$  in  $\mathbf{P}^{n*}$ . Let  $\omega$  be any holomorphic differential on  $C$ . By the differentiated form of Abel's theorem,  $\sum \omega(p_i(\xi)) = 0$ , where each  $\omega(p_i(\xi))$  is a one-form on  $U$  of the form  $a(p_i)dp_i$ . If one now uses local coordinates on  $C$  and lets  $u_i(\xi)$  be the local coordinate expression for  $p_i(\xi)$ , then the hyperplanes  $u_i = \text{constant}$  define the linear  $d$ -web on  $U$  and the web normals satisfy an equation (1)  $\sum f_i(u_i)du_i = 0$ .

For an arbitrary  $d$ -web choose functions  $u_1, \dots, u_d$  such that  $u_i = \text{constant}$  gives the leaves of the  $i$ th foliation. An equation of the form (1) is called an abelian equation. The rank of a  $d$ -web is defined to be the number of linearly independent abelian equations. This rank does not depend on the choice of defining functions and is an invariant of the web. Returning to the algebraic curve  $C$ , one has that the rank of the associated web is the genus of  $C$ . Castelnuovo in 1889 determined a sharp bound for the genus of a nondegenerate algebraic curve of degree  $d$  in  $\mathbf{P}^n$ . Call his bound  $\pi(n, d)$ . Thus, there are  $d$ -webs on  $\mathbf{R}^n$  of rank  $\pi(n, d)$  and such webs are algebraic in the sense that they are constructed from algebraic curves in  $\mathbf{P}^n$ . Now the first author, as part of his dissertation written under Blaschke, proved that this same number serves as a bound for the rank of any  $d$ -web. Thus, it is natural to investigate if a  $d$ -web of maximum rank is necessarily algebraic.

The main result of this paper is that a  $d$ -web of maximum rank is linearizable. But by a result of the second author [*Invent. Math.* **35** (1976), 321–390; MR0435074 (55 #8036)] a linear web which admits at least one abelian equation is algebraic. Thus, a  $d$ -web of maximum rank is equivalent to the web defined by one of Castelnuovo's extremal curves. The problem of finding invariants which describe when a web is linearizable, but without requiring a large rank, is thought to be more difficult.

We shall briefly outline the proof of the main result. This proof is in three stages. The first uses some ideas of algebraic geometry to define a system of curves on  $U$ . The second shows that these curves form a path geometry in the sense of classical projective differential geometry. In the last stage the projective curvature

is shown to vanish. So the curves can be transformed into straight lines and this transformation takes the leaves of the web to hyperplanes.

First stage: Recall that to a compact algebraic curve  $C$  of genus  $r$  there is an associated canonical curve  $F(C)$  in  $\mathbf{P}^{r-1}$  defined by any choice of a basis for the holomorphic differential forms on  $C$ . It follows from Abel's theorem that the curve  $F(C)$  is highly nongeneric—if  $C$  is a nondegenerate curve of degree  $d$  in  $\mathbf{P}^n$  and  $\xi \in \mathbf{P}^{n*}$  then the points  $F(p_1(\xi)), \dots, F(p_d(\xi))$  span some  $\mathbf{P}^{d-n-1}$ . An analogous construction for any web of maximal rank  $r$  yields a map  $U \rightarrow \mathbf{P}^{r-1}$ ,  $x \mapsto [z_1(x), \dots, z_r(x)]$ , such that for each  $x$  the points  $\{z_1(x), \dots, z_r(x)\}$  all lie on a rational normal curve  $E_x$  in some  $\mathbf{P}^{d-n-1}$ . Also, the normals to the leaves at  $x$  lie on a rational normal curve  $D_x$  in  $\mathbf{P}(T_x^*)$  and there is a natural projectivity from  $D_x$  to  $E_x$ . Note that the curves  $D_x$  give a distribution of curves over  $U$  while all the curves  $E_x$  lie in a given space  $\mathbf{P}^{r-1}$ . A system of curves  $x(t)$  is defined on  $U$  by the condition that infinitely near curves in  $E_{x(t)}$  intersect in  $n-1$  points. Each point in  $D_x$  corresponds to a normal to a hypersurface in  $U$  with the property that any such curve tangent to the hypersurface at one point must always remain in the hypersurface. In this way one obtains a two-parameter family of hypersurfaces which includes the leaves of the web.

Second stage: The system of curves  $x(t)$  is in fact defined by an overdetermined system of second order ordinary differential equations. To show that these curves actually exist one must study compatibility conditions. The key idea is to consider the  $G$ -structure defined by the distribution of the normal curves  $D_x$  and use the associated moving frames as a tool in the compatibility calculations. The curves  $x(t)$  thus are shown to form a path geometry. Each hypersurface in the above two-parameter family is totally geodesic in this path geometry.

Third stage: The problem is now one of projective differential geometry. To any path geometry may be associated a unique normalized projective connection. The curvature of this connection is identically zero precisely when there is a diffeomorphism which maps each curve to a straight line (and thus each totally geodesic hypersurface to a hyperplane). Finally, a very nice result completes the proof: If in any path geometry we are given a distribution of rational normal curves  $D_x \subset \mathbf{P}(T_x^*)$  such that each  $\omega \in D_x$  is the normal to a totally geodesic hypersurface through  $x$ , then the associated normalized connection has zero curvature.

Many various techniques are used in this proof but all are carefully motivated and explained and the paper is self-contained to a remarkable degree. Among the many digressions and explanations, the interested reader may find Steiner's synthetic construction of the conic through five points and various generalizations; a derivation of Castelnuovo's bound for the genus and Chern's bound for the rank; the relation between Abel's theorem and the Riemann-Roch theorem; a discussion of projective differential geometry and the proof that there is a unique projective connection associated to any path geometry; and a proof of Beltrami's theorem that the geodesics of a Riemannian metric form a flat path geometry if and only if the metric has constant sectional curvature.

The classical theory of  $d$ -webs is by definition the content of the book by W. Blaschke and G. Bol [*Geometrie der Gewebe*, Springer, Berlin, 1938; Jbuch **20**, 67]. Similar material is in Blaschke's lecture notes ["Lectures on topological questions in differential geometry", delivered at the University of Chicago, Chicago, Ill., 1932]. An expository paper closely related to the reviewed paper is by the second author [*Algebraic geometry* (J. J. Sylvester Sympos., Johns Hopkins Univ., Baltimore, Md.,

1976), pp. 26–51, Johns Hopkins Univ. Press, Baltimore, Md., 1977; MR0480492 (58 #655)]. Finally, the authors have begun to study similar questions for webs of codimension greater than one. In another paper [53009 below], they define the rank of such a web and find similar sharp bounds.

From MathSciNet, December 2008

*H. Jacobowitz*