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The maximum principle, by Patrizia Pucci and James Serrin, Progress in Nonlinear Differential Equations and Their Applications, 73, Birkhäuser Verlag, Basel, 2007, x+235 pp., US $64.95, ISBN-13: 978-3-7643-8144-8

1. Introduction

Maximum principles are among the most powerful and widely used analytic tools in the study of second-order linear and nonlinear elliptic and parabolic equations. They enable us to obtain valuable information about (real valued) solutions of differential equations and inequalities (such as a priori pointwise estimates, and uniqueness and stability results) without the need to know in advance the solutions explicitly, or without even knowing a priori the existence of such solutions. As a matter of fact, in many cases, the maximum principle (or MP for brevity) is an essential ingredient in proving also existence theorems. Moreover, MPs are closely related to some well known important qualitative properties of solutions of such equations, e.g., Harnack inequalities, comparison principles and tangency theorems, Phragmèn-Lindelöf principles, removability of isolated singularities, and Liouville theorems.

As an illustration of the MP, we mention the everyday fact that a body with a prescribed (time-independent) boundary temperature attains the highest temperature of a steady-state temperature on its boundary.

The present article briefly surveys the exciting field of maximum principles with emphasis on the content and features of the book under review (which will be denoted by [PS]).

2. What are MPs all about

A familiar heuristic principle in the study of elliptic equations states that every property of the Laplace operator remains true for general second-order elliptic operators if properly formulated. Following this rule of thumb, we shall present in this section a few features of the MP which can be quite easily proved for the Laplacian using only advanced calculus arguments. Surprisingly, the proofs of the analogous results for general linear and nonlinear elliptic and parabolic operators are in many cases based on the same considerations. Generally speaking, the maximum principle for such operators relies mainly on the ellipticity of the operator and not on other properties (such as the smoothness of the coefficients or the structure of the equation).

Throughout this review, D denotes a domain in $\mathbb{R}^d$. The Laplace operator is defined on (real valued) functions $u \in C^2$ by

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2 u(x)}{\partial x_i^2}.$$

A function $u \in C^2(D)$ satisfying in D the equation $\Delta u = 0$ (resp. the differential inequality $\Delta u \geq 0$) is called *harmonic* (resp. *subharmonic*) in D. The function u is *superharmonic* in D if $-u$ is subharmonic in D.

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Note that for $d = 1$, a subharmonic function is a convex function. It is well known that a *nonconstant* C^2-convex function u defined on an interval $[a, b] \subset \mathbb{R}$ satisfies a strong MP. Namely, the maximum of u is attained *only* at the boundary. Moreover, the one-sided first derivative of u at a maximum point is not zero (end point lemma). The higher dimension generalizations are given in the following theorems.

Theorem 2.1 (Weak maximum principle)**.** *Let $D \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain, and let $u \in C^2(D) \cap C(\bar{D})$ be a subharmonic function in D. Then the maximum of u in $\bar{D}$ is achieved on the boundary ∂D.*

Proof. If $v \in C^2(D) \cap C(\bar{D})$ satisfies $\Delta v > 0$ in D, then v cannot have a local maximum point in D. Indeed, if $x_0 \in D$ is a local maximum point of v, then $\Delta v(x_0) \leq 0$, in contradiction to our assumption.

Notice that for any $\varepsilon > 0$, the function $v_\varepsilon(x) := u(x) + \varepsilon|x|^2 = u(x) + \varepsilon \sum_{i=1}^d x_i^2$ satisfies $\Delta v_\varepsilon \geq 2\varepsilon d > 0$. Set $M := \max_{\partial D} u$, and $L := \max_{\partial D} |x|^2$. From our argument applied to v_ε, it follows that $v_\varepsilon \leq M + \varepsilon L$ in D. Since $u = v_\varepsilon - \varepsilon|x|^2$, it now follows that $u \leq M + \varepsilon L$ in D. Because ε is arbitrary, we obtain $u \leq M$ in D. □

Remark 2.2. For any set A and for any function u, we have $\min_A u = -\max_A(-u)$. It follows that a superharmonic function $u \in C^2(D) \cap C(\bar{D})$ satisfies the *weak minimum principle* in a bounded domain D. Namely,

$$\min_{x \in \bar{D}} u(x) = \min_{x \in \partial D} u(x).$$

The weak maximum principle (or WMP for brevity) readily implies the uniqueness and stability of solutions to the Dirichlet problem:

Corollary 2.3. *Consider the Dirichlet problem in a bounded domain D:*

$$\begin{cases} \Delta u = f, & x \in D, \\ u(x) = g(x), & x \in \partial D. \end{cases} \tag{2.1}$$

1. Problem (2.1) *has at most one solution in $C^2(D) \cap C(\bar{D})$.*

2. Let $u_1, u_2 \in C^2(D) \cap C(\bar{D})$ be solutions of the Poisson equation $\Delta u = f$ with the Dirichlet continuous boundary data g_1 and g_2, respectively. Then

$$\max_{x \in D} |u_1(x) - u_2(x)| \leq \max_{x \in \partial D} |g_1(x) - g_2(x)|.$$

The boundedness of D is essential. Indeed, for $d = 2$, the functions $u_1 = \mathbf{0}$ and $u_2(x) = \log|x|$ both solve the Dirichlet problem

$$\begin{cases} \Delta u = 0, & |x|^2 > 1, \\ u(x) = 0, & |x|^2 = 1. \end{cases}$$

We note that if ∂D is smooth and $u \in C^1(\bar{D})$ is any function attaining its maximum at some boundary point $x_0 \in \partial D$, then $\partial u/\partial \nu \geq 0$ at x_0 for any vector ν which points outward from D at the point x_0. It turns out that if u in addition is a nonconstant subharmonic function in D, then a strict inequality holds, provided x_0 satisfies the *interior sphere condition* at x_0. We have:

Theorem 2.4 (Boundary point lemma)**.** *Let $u \in C^2(D) \cap C(\bar{D})$ be a subharmonic function in a domain D. Suppose that $x_0 \in \partial D$ lies on the boundary of an open* *ball $B(y,R) \subset D$, and that an outward directional derivative $\partial u/\partial \nu$ exists at x_0. If $u(x) < u(x_0)$ in D, then*

$$\frac{\partial u}{\partial \nu} > 0 \qquad \text{at } x_0.$$

Proof. It can be verified that for any $\alpha > d/2$, the function

$$v(x) := \mathrm{e}^{-\alpha|x-y|^2} - \mathrm{e}^{-\alpha R^2}$$

is a nonnegative subharmonic function in $B(y,R)$ that vanishes on $\partial B(y,R)$ and satisfies $\partial v/\partial \nu < 0$ at x_0. Since $u(x) - u(x_0)$ is strictly negative in $B(y,R)$, it follows that for some $\varepsilon > 0$, and $0 < \rho < R$ the function $w(x) := u(x) - u(x_0) + \varepsilon v(x) \leq 0$ on $\partial B(y,R) \cup \partial B(y,\rho)$. By the WMP $w(x) \leq 0$ in $B(y,R) \setminus B(y,\rho)$, and w attains its maximum at x_0. Therefore,

$$\frac{\partial w}{\partial \nu} \geq 0 \quad \text{at } x_0 \quad \Rightarrow \quad \frac{\partial u}{\partial \nu} \geq -\varepsilon \frac{\partial v}{\partial \nu} > 0 \quad \text{at } x_0. \qquad \square$$

The boundary point lemma for elliptic operators is due to E. Hopf and O. A. Oleinik. Using the WMP and the boundary point lemma, we obtain the following uniqueness theorem for the Neumann problem.

Corollary 2.5. *Let D be a smooth bounded domain. Then up to an additive constant, the Neumann problem*

$$\begin{cases} \Delta u = f, & x \in D, \\ \dfrac{\partial u}{\partial n} = g, & x \in \partial D, \end{cases} \qquad u \in C^2(D) \cap C^1(\bar{D}), \tag{2.2}$$

has at most one solution. Here $\partial/\partial n$ is the exterior normal derivative.

The WMP still does not exclude the possibility that the maximum of a subharmonic function is attained also at an internal point. The boundary point lemma implies the following strengthening of Theorem 2.1.

Theorem 2.6 (Strong maximum principle). *Let D be any domain in $\mathbb{R}^d$, and let $u \in C^2(D)$ be a nonconstant subharmonic function in D. Then the maximum of u cannot be attained at any interior point of D.*

The above MP is called "strong" since there are no assumptions on the domain D nor on the behavior of u at the boundary. The strong maximum principle (or SMP for brevity) implies the following comparison principle.

Corollary 2.7 (Strong comparison principle). *Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $u_i \in C^2(\Omega) \cap C(\bar{D})$, $i = 1, 2$, satisfy the following inequalities:*

$$\begin{cases} -\Delta u_1 \leq -\Delta u_2, & \text{in } D, \\ \quad u_1 \leq u_2, & \text{on } \partial D. \end{cases} \tag{2.3}$$

Then $u_1 < u_2$ in D unless $u_1 = u_2$ in D.

Degenerate nonlinear elliptic operators usually satisfy only a weak comparison principle (one might conclude only that $u_1 \leq u_2$ in D), and one should impose further conditions for the validity of the strong comparison principle which is called in [PS] the *tangency theorem* (see Chapter 2).

The question of the existence of a solution to the Dirichlet problem (2.1) is frequently treated by using *Perron's method*, which is heavily based on the MP, and can be easily extended to general second-order linear and nonlinear elliptic and parabolic equations. Denote by $\mathcal{S}_g$ the set of all subharmonic functions $v \in C(\bar{D})$ satisfying $v \leq g$ on ∂D. *Perron's solution* u of the Dirichlet problem (2.1) in a bounded domain D with $f = 0$ and $g \in C(\partial D)$ is defined by

$$u(x) = \sup_{v \in \mathcal{S}_g} v(x), \qquad x \in D.$$

It follows that u is a well defined *smooth* harmonic function in D. Moreover, if D is smooth enough, then $u \in C(\bar{D})$ and $u = g$ on ∂D.

Recall that a boundary value problem is said to be *well-posed* if it has a unique solution that continuously depends on the sources, including boundary conditions. Using the results obtained, we see that the validity of the MP implies the well-posedness of the classical Dirichlet problem. This statement remains true for general linear and nonlinear second-order elliptic operators.

3. MPs for linear elliptic operators

Let L be a second-order linear elliptic operator of the form

$$L(x, \partial) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x), \qquad x \in D, \tag{3.1}$$

where $\partial_i = \partial/\partial x_i$. We assume that L is strictly *elliptic*; that is, for each $x \in D$ the real (symmetric) matrix $[a_{ij}(x)_{ij}]$ is positive definite on $\mathbb{R}^d$. Suppose that the coefficients of L are smooth enough in D. A function $u \in C^2(D)$ is a *subsolution* (resp., *supersolution*) of L in D if $Lu \geq 0$ (resp., $Lu \leq 0$) in D.

It turns out that slightly modified weak and strong MPs for subsolutions, as well as a boundary point lemma and a comparison principle, hold true in any bounded subdomain D' satisfying $\overline{D'} \subset D$ if and only if the operator L admits a positive supersolution (or solution) of the equation $Lu = 0$ in D (it seems that this important fact is not mentioned in [PS]). The proofs for the linear case are similar to Hopf's technique presented above for the Laplacian. Moreover, this statement (if properly formulated) is also true for the class of weak solutions of elliptic equations in divergence form as well as for the class of strong solutions of strongly elliptic equations with locally bounded coefficients. The existence of a global positive supersolution can be verified easily if $c \leq 0$ or if D is a "thin" domain (see, e.g., [1, 3] and Section 3.3).

4. MPs for nonlinear elliptic operators

The book under review considers substantially a very general class of quasilinear equations in divergence form $\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0$, where A and B satisfy growth conditions compatible with the Sobolev space $W^{1,p}$, $1 < p < \infty$. The equations considered include the celebrated p-Laplacian, the mean curvature operator, and similar equations.

In fact, using a standard linearization procedure, E. Hopf proved under appropriate assumptions weak comparison principles and tangency theorems for *fully nonlinear* second-order elliptic equations of the form

$$\mathcal{F}(x, u, Du, D^2u) = 0, \qquad x \in D. \tag{4.1}$$

If one further assumes that the given operator has an additional structure (e.g., semilinear operator of certain type, quasilinear operator in divergence form satisfying some structured elliptic inequalities, homogeneity assumptions, etc.), one obtains MPs and comparison principles, as well as corresponding boundary point lemmas under less restrictive assumptions. These results are based on the celebrated work of Serrin [5], and are discussed in detail in Chapters 3 and 6 of [PS].

5. The compact support principle

Let $f \in C(\mathbb{R})$ be a nondecreasing function satisfying $f(0) = 0$. Let $F(u) := \int_0^u f(s)\,\mathrm{d}s$. Assume that $u \geq 0$ satisfies the semilinear differential inequality $\Delta u - f(u) \leq 0$ in a domain $D \subset \mathbb{R}^d$. If $\int_0^1 (\sqrt{F(s)})^{-1}\,\mathrm{d}s = \infty$, then the SMP holds true. That is, if u vanishes at some point of D, then u must vanish everywhere in D.

On the other hand, if $\int_0^1 (\sqrt{F(s)})^{-1}\,\mathrm{d}s < \infty$, and $u \geq 0$ satisfy the differential inequality $\Delta u - f(u) \geq 0$ in an exterior domain $D \subset \mathbb{R}^d$, and $u(x) \to 0$ as $|x| \to \infty$, then u has a compact support. This phenomenon is often called the *compact support* or the *dead core principle*.

In Chapter 5 of the book under review, the authors present generalizations of the above phenomena to the case of (degenerate) quasilinear operators in divergence form. In particular, the above two results hold true for the p-Laplacian and the mean curvature operators.

6. Harnack inequality and Hölder continuity

The maximum principles usually give us *global* bounds. On the other hand, *local* pointwise estimates and regularity theory are fundamental tools in the study of quasilinear equations. The most important results in these directions are Harnack inequalities for nonnegative solutions, and the Hölder continuity of weak solutions. In Chapter 8 of [PS], the authors present in a very clear way detailed proofs of these results for general quasilinear equations based on Moser's iteration technique and Serrin's seminal work [5].

7. Liouville theorems

The classical Liouville theorem asserts that any bounded (or nonnegative) entire harmonic function in $\mathbb{R}^d$ must be a constant. This statement is also true for solutions of the equation $Lu = 0$ in $\mathbb{R}^d$ provided that L is either of the form (3.1), and the coefficients of L converge sufficiently fast (as $|x| \to \infty$) to the coefficients of the Laplacian, or L is a second-order linear uniformly elliptic operator in divergence form with bounded measurable coefficients. Theorems of this kind are known as Liouville or Cauchy-Liouville theorems. In Chapter 8 the authors present among some other applications some Cauchy-Liouville theorems for quasilinear equations.

8. Conclusions

The book under review is written by leading experts who have made extensive and deep contributions to the subject for many years; indeed, a substantial part of the book's material is due to the authors' research. The book is an in-depth, up-to-date, modern, clear exposition of the advanced theory of MPs. There are a number of excellent classical books discussing mainly the MPs for linear elliptic operators (see [2, 3, 4]). They should be considered as a good and essential starting point before studying the book under review which treats the subject for very general and different frameworks. On the other hand, the prerequisites assumed do not go much beyond a first course in analysis and functional analysis, and the proofs of the main theorems are entirely self-contained. Therefore, the book should be accessible to a large audience including graduate and postgraduate students and researchers in the field of partial differential equations.

Each chapter is followed by notes that provide a short historical background, and by exercises, some of which contain new results. The connections among different topics are clearly exhibited. Much attention is given to bibliographical and historical notes.

Many topics appear in this volume for the first time in book form, e.g., the compact support principle, Cauchy-Liouville theorems for quasilinear equations, and nonhomogeneous structured elliptic inequalities. On the other hand, the book omits many topics related to MPs, e.g., the Alexandrov-Bakelman-Pucci weak maximum principle, the principal eigenvalue and ground state (and in particular the important work [1]), criticality theory, relations to spectral theory, removable singularities, and the anti-maximum principle.

Regardless of this remark, I strongly recommend this excellent book to every researcher or graduate student in the field of elliptic equations. Naturally, it will also be of interest to many mathematicians in related areas.

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Yehuda Pinchover

Technion - Israel Institute of Technology

E-mail address: `pincho@techunix.technion.ac.il`