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*The 1-2-3 of modular forms*, by J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier, Universitext, Springer-Verlag, Berlin, Heidelberg, 2008, x+266 pp., softcover, US \$69.95, ISBN 978-3-540-74117-6

## 1. Introduction

The term *modular form*, generically, is very broad. These objects most naturally exist within the scope of number theory and play central roles in its various branches, yet they also play important roles in other fields of mathematics. For example, modular forms might conjure up such diverse thoughts as Fermat's Last Theorem, the Langlands program, the Riemann Hypothesis, arithmetic applications or geometric interpretations, $L$-functions and elliptic curves, applications to string theory, combinatorics, or cryptography. Of course this is only a small sampling of the roles in which modular forms exist and play predominant roles; there are a myriad of others that keep modular forms a very active area of research. A standard way to define these pervasive objects, in the classical sense, is as a complex analytic function $f : H := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \to \mathbb{C}$ equipped with a certain symmetry under the action of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ (or a subgroup). That is, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. The origins of this definition date back to the first half of the nineteenth century, to the era of Jacobi and Eisenstein. Since then, many generalizations have been defined and studied, and in tandem with classical modular forms, they weave through many of the impressions mentioned above that modular forms have made.

Theta functions are some of the most fundamental examples of modular forms, both classically and more generally, and are arguably as ubiquitous. Table 1 gives a list of various theta functions that have emerged over the centuries.

Even a person who has not studied the functions of Table 1 previously might agree that some sort of an evolution occurs between the first and last entries. Indeed, *The 1-2-3 of Modular Forms* will indirectly attest to this through the expositions of its authors, each a leading expert in a related area of mathematics chosen to give a series of lectures at the "Modular Forms and Their Applications" summer school of June 2004, at the Sophus Lie Conference Center in Nordfjordeid, Norway. In the sections that follow, I'll elaborate in more detail on certain roles modular forms have played throughout history, emphasizing both classical and contemporary problems and applications. The more serious reader is pointed toward the book under review for the experts' accounts, which are very thorough and extremely well written lecture series on elliptic, Hilbert, and Siegel modular forms and their applications.

## 2. Classical modular forms

Consider the Jacobi theta function of Table 1. Raising this function to the $r$th power and expanding, one can see that

$$(1)\qquad \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^r = \sum_{n_1,n_2,\ldots,n_r} q^{n_1^2+n_2^2+\cdots+n_r^2} = \sum_{n\geq 0} s_r(n)q^n,$$

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TABLE 1. Various theta functions

| | |
|---|---|
| $\sum_{n\in\mathbb{Z}} q^{n^2}$ | Jacobi $\theta$-function |
| $\sum_{\vec{x}\in\mathbb{Z}^n} q^{Q(\vec{x})}$ | Multivariable $\theta$-function |
| $\sum_{A\in\mathbb{Z}^{r\times g}} \rho(\sqrt{B}A)e\left(\frac{1}{2}\mathrm{Tr}(A^tBA\tau)\right)$ | Siegel $\theta$-function |
| $\sum_{\lambda\in L^\vee} \frac{(\lambda,Z)^r}{(Z,\overline{Z})^r}e(Q(\lambda_Z)N\tau+Q(\lambda_Z)N\overline{\tau}))$ | $\theta$-kernel |

where $s_r(n)$ counts the number of ways to represent $n$ as a sum of $r$ squares. Arithmetic generating functions, such as the right-most series displayed in (1), were historically influential in the development of the study of modular forms. Yet how were such *counting functions* related to complex analytic functions with certain symmetries? If one replaces $q$ with the function $e(z) := e^{2\pi iz}, z \in \mathbb{H}$, then the Jacobi $\theta$-function becomes a function on $\mathbb{H}$. Moreover, in this case, an application of the *Poisson summation formula*, which relates a periodic function and its Fourier transform, will show that under a suitable change of varable, this Jacobi $\theta$-function transforms appropriately under $\Gamma$.

The story of modular forms, however, began with an earlier relative to Jacobi's $\theta$-functions, namely with elliptic functions, which are doubly periodic meromorphic complex functions. Elliptic functions were studied by Weierstrass and date back to Gauss, and they led naturally to the study of elliptic curves, which are intimately related to modular forms. In general, a typical elliptic curve $E$ over a field $K$ might be defined by solutions $(x,y)$ to the equation $E: y^2 = ax^3+bx+c$. Over the field $\mathbb{C}$, Weierstrass made an important analytic connection, showing that a certain lattice sum $\wp(z)$ defines points on an elliptic curve $(\wp(z),\wp'(z))$ when one takes $a = 4$, $b = -120G_4(z)$ and $c = -280G_6(z)$, and the functions $G_k(z)$ are the *Eisenstein series*

$$G_k(z) := \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}(mz+n)^{-k},$$

where $k > 2, z \in \mathbb{H}$. It is not difficult to see that the specialization of the Riemann $\zeta$-function, $\zeta(s) := \sum_{n\geq 1} n^{-s}$, at $s = k$ is a factor of the Eisenstein series: $G_k(z) = \zeta(k)E_k(z)$. Moreover, the factors $E_k(z)$ also play important roles in the theory of modular forms.

**Theorem.** *The Eisenstein series $E_4(z)$ and $E_6(z)$ "generate" the ring of modular forms on* $\mathrm{SL}_2(\mathbb{Z})$*.*

The Riemann $\zeta$-function itself, as a function in the complex variable $s$, is one of the most central objects in number theory. While defined only for $\mathrm{Re}(s) > 1$, Riemann showed that $\zeta(s)$ can be analytically continued, and that the following functional equation holds:

$$\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s),$$

where $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$, $\sigma > 0$, and $s := \sigma + it$. The famous Riemann Hypothesis, still unsolved, is as follows.

**The Riemann Hypothesis.** If $s = \sigma + it$ is such that $\zeta(s) = 0$, then $\sigma = 1/2$.

There is of course a deeper relationship between the $\zeta$-function and modular forms than a specialization appearing as a factor of the Eisenstein series. The Riemann $\zeta$-function is an example of an *$L$-series*, an object which may be associated to both modular forms and elliptic curves, and which also provides a beautiful connection between them.

An elliptic curve defined over $\mathbb{Q}$ has a natural structre as an abelian group, and Mordell's theorem asserts that $E(\mathbb{Q}) \cong \mathbb{Z}^r \bigoplus E(\mathbb{Q})^{\text{tors}}$, where the number $r$ is called the *rank* of $E$. One can study $E$ by looking at its points over finite fields $\mathbb{F}_p$ of prime order $p$. If $a_p := p + 1 - \#\{\text{points of } E \text{ over } \mathbb{F}_p\}$, one can form an $L$-function associated to $E$ by $L(s,E) := \prod_{\text{primes } p} L_p(s,E)$ where the local factors $L_p(s,E)$ are given by either $(1 - a_p p^{-s} + p^{1-2s})^{-1}$ or $(1 - a_p p^{-s})^{-1}$, depending on certain divisibility condtions on $p$. The famous Birch and Swinnerton-Dyer conjecture makes a striking connection between an analytic object, the $L$-function of an elliptic curve, and an algebraic one, its rank.

**Conjecture** (Birch and Swinnerton-Dyer)**.** *$L(E,s)$ has a zero at $s = 1$ of order $r$.*

A natural object to study in light of connections between the elliptic and modular theories is the series $\sum_n a_n q^n$, which one would hope to be a modular form. The Taniyama-Shimura conjecture remarkably asserts that *all* elliptic curves over $\mathbb{Q}$ are modular (meaning the associated series $\sum_n a_n q^n$ is a modular form), and is required to make sense of the meaning of the value of $L(E,s)$ at $s = 1$. The proof of the Taniyama-Shimura conjecture is due to Andrew Wiles, and was an integral part of the proof of Fermat's Last Theorem.

To futher illustrate the birth of modular forms from the elliptic theory, one has the *modular invariant*

$$j(z) := 1728\frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2},$$

which defines an isomorphism

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \xrightarrow{j} \mathbb{C}.$$

The surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is realized by identifying points $z_1, z_2 \in \mathbb{H}$ such that $\gamma z_1 = z_2$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Or, if one prefers the language of elliptic curves, $j$ can be thought of as providing a bijection between $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and elliptic curves over $\mathbb{C}$ (modulo equivalence), where one uses the fact that any elliptic curve over the field $\mathbb{C}$ can be represented as a quotient $\mathbb{C}/\Lambda$ for some complex lattice $\Lambda$. Understanding the values of the $j$-function at certain imaginary quadratic points in $\mathbb{H}$, called *CM points*, has also been an important question in number theory. These points are defined as follows. Given an elliptic curve $E$ over $\mathbb{C}$ corresponding to a lattice $\Lambda$, if there exists some $\lambda \in \mathbb{C} - \mathbb{R}$ such that $\lambda\Lambda \subseteq \Lambda$, then the elliptic curve $E$ is said to admit *complex multiplication*. Points $z \in \mathbb{H}$ are called *CM points* if the corresponding elliptic curve (under the bijection mentioned above) has complex multiplication.

What can be said about *singular moduli*, these specializations of the $j$-function at CM points? Kronecker's so-called *Jugendtraum* was the hope of finding functions whose special values would generate Abelian extensions of arbitrary number fields, paralleling the role of the exponential function over $\mathbb{Q}$. This problem is still unsolved in full generality; however, Kronecker did succeed in establishing that values of the $j$-function generate Abelian extensions of imaginary quadratic fields.

B. Gross and D. Zagier's famous study of heights of Heegner points [9] led them to completely determine the factorization of singular moduli, in itself a highly celebrated result. For natural reasons, they were led to consider differences of singular moduli and proved the following result, a special case of which yields explicit expressions for singular moduli.

**Theorem** (Gross-Zagier)**.** *Let $K$ and $k$ be imaginary quadratic fields with relatively prime discriminants $D$ and $d$, respectively. Then*

$$\prod_{\substack{z\in \mathrm{CM}_D\\ \omega\in\mathrm{CM}_d}} (j(z)-j(\omega))^{8/Ww} = \pm \prod_{\substack{x\in\mathbb{Z},n,m\in\mathbb{Z}^+\\ x^2+4nm=dD}} n^{\epsilon(m)},$$

*where $W$ (resp. $w$) is the number of units of $K$ (resp. $k$), $\mathrm{CM}_D$ (resp. $\mathrm{CM}_d$) is the set of equivalence classes of points $z$ (resp. $\omega$) corresponding to the elliptic curves $\mathbb{C}/\mathbb{Z}z+\mathbb{Z}$ (resp. $\mathbb{C}/\mathbb{Z}\omega+\mathbb{Z}$) with complex multiplication by $\mathcal{O}_K$ (resp. $\mathcal{O}_k$), and $\epsilon$ is an explicitly defined genus character.*

## 3. Generalized modular forms

The Gross-Zagier theorem is an example of a theorem that exists as stated within the scope of the classical theory, but which admits a natural generalization to other settings (recently proved by Bruiner-Yang [5] and described below). As examples of more general modular forms, consider the last two theta functions of Table 1. Naturally, one could imagine replacing $\mathrm{SL}_2(\mathbb{Z})$ in the classical setting by another matrix group, or considering forms not just of one variable $z\in\mathbb{H}$, but more than one. In Table 2 we exhibit such analogies and depict some tools used to build other types of modular forms.

In a special case of the Hilbert modular setting, one considers a real quadratic field $F$, and defines the action of $\mathrm{SL}_2(F)$ on $\mathbb{H}\times\mathbb{H}$ by $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) z = \left(\frac{az_1+b}{cz_1+d}, \frac{a'z_2+b'}{c'z_2+d'}\right)$, where $x'$ denotes the conjugate of $x$. Further, as alluded to in Table 2, in the Siegel modular setting the upper-half complex plane $\mathbb{H}$ is replaced by the Siegel upper-half plane $\mathcal{H}_g$ of genus $g$, consisting of all $g\times g$ symmetric matrices over $\mathbb{C}$ with positive definite imaginary part. Thus, $\mathbb{H}=\mathcal{H}_1$. The elements of the automorphism group $\mathrm{Sp}_{2g}(\mathbb{Z})$ can be written as $\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)$ where $A,B,C,D$ are $g\times g$ integer matrices satisfying a set of relations generalizing those defining $\mathrm{SL}_2(\mathbb{Z})$.

Various analogies exist between the theories; all three types of modular forms mentioned here have certain Fourier series expansions, while certain other properties in the classical setting are perturbed. For example, the surfaces $\Gamma\backslash\mathbb{H}$, $\Gamma_F\backslash\mathbb{H}\times\mathbb{H}$ or $\Gamma_g\backslash\mathcal{H}_g$ are not compact, yet the surfaces can be made compact. In the classical and Hilbert settings, this is done by adjoining a set of points called *cusps*. One difference between the classical setting and the Hilbert modular setting is that a holomorphic Hilbert modular form will necessarily be holomorphic at the cusps, a fact due to Göetzky-Koecher.

TABLE 2. Tools for building modular forms

| **Classical** | **Hilbert** | **Siegel** |
|---|---|---|
| $\mathbb{Q}$ | $F := \mathbb{Q}(\sqrt{d}), d \in \mathbb{N}$ | $\mathbb{Q}$ |
| $\mathrm{SL}_2(\mathbb{Z}) = \Gamma$ | $\mathrm{SL}_2(\mathcal{O}_F) = \Gamma_F$ | $\mathrm{Sp}_{2g}(\mathbb{Z}) = \Gamma_g$ |
| $\mathbb{H}$ | $\mathbb{H} \times \mathbb{H}$ | $\mathcal{H}_g := \left\{\gamma \in \mathrm{M}_2(\mathbb{C}) \middle\vert \, {}^{\mathrm{Im}(\gamma)>0}_{\gamma^t=\gamma}\right\}$ |
| $f : \mathbb{H} \to \mathbb{C}$ | $f : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ | $f : \mathcal{H}_g \to V$ |
| $f(\gamma z) = (cz+d)^k f(z)$ | $f(\gamma z) = (cz_1+d)^{k_1}(c'z_2+d')^{k_2} f(z)$ | $f(\gamma z) = \rho(Cz+D)f(z)$ |
| $k \in \mathbb{Z}$ | $(k_1, k_2) \in \mathbb{Z}^2$ | $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}(V)$ |

Analogies in the Hilbert and Siegel modular settings that do carry over from the classical setting, for example, are notions of Eisenstein series, which generate spaces of modular forms, as well as $L$-functions and theta functions. Interplay between spaces of different types of modular forms exists as well. The Siegel *theta kernel* of Table 1, for example, is used by Borcherds [3] to *lift* modular forms $f$ of level $N$ to modular forms on another group $O(L)$ called the *orthogonal group* of the lattice $L$, where one may regard the Hilbert modular group situated via a certain isomorphism. Such a *lift* looks as follows:

$$\int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{\Theta_r(z,Z)}v^{k-2}dudv,$$

where $\Theta_r(z,Z)$ is the theta kernel in Table 1.

Furthering Borcherds' theory, recently Bruiner and Yang have generalized the work of Gross and Zagier on CM and singular moduli to the Hilbert modular setting [5]. One of Borcherds' results is that the Hilbert modular form associated to $f$ has a product expansion, and looks as follows:

$$\Psi(z,f) = q_1^{\rho}q_2^{\rho'} \prod_{\substack{\nu \in \mathfrak{d}_F^{-1} \\ (\nu,W)>0}} (1 - q_1^{\nu}q_2^{\nu'})^{\tilde{c}(p\nu\nu')}.$$

Here, the sum ranges over certain elements in the inverse different $\mathfrak{d}_F^{-1}$ of the field $F$ (and $\mathfrak{d}$ is the ideal generated by the square root of the field discriminant). The objects $W$ and $\rho$ appearing in the Borcherds' product above are what are called the "Weyl chamber" and "Weyl vector", respectively, associated to the form $f$, and the interested reader can find a detailed definition in the book under review. To consider an analogue of Gross-Zagier, one must first formulate the notion of a CM point in the Hilbert modular setting. Such a definition exists, and the product of a function $f$ over such CM-points (a CM-cycle) is written as $f(\mathcal{CM}(K))$, where $K$ is the associated imaginary quadratic field extension of $F$. The result of Bruinier-Yang generalizing that of Gross-Zagier is as follows.

**Theorem** (Bruinier-Yang). *With notation as above,*

$$\Psi(\mathcal{CM}(K)) = \pm \prod_{\ell \text{ rational prime}} \ell^{e_\ell},$$

*where $e_\ell$ are explicitly given constants.*

In general, lifts have been important in the theory of modular forms; without a bijection between spaces of modular forms, one naturally obtains spaces of forms begging to be understood, and bijections are of course of interest. Classically, *Hecke operators* $T(p)$ for primes $p$ act on modular forms $f(z) = \sum_n a_n q^n$, and common eigenforms (meaning $f|T(p) = \lambda(p)f$) of the Hecke operators have eigenvalues $\lambda(p)$ given by their Fourier coefficients $a_p$. Shimura [14] showed that the classical theory could be generalized to higher genus $g > 1$, and this was done in [1], [2], and [8]. Regarding lifts, the Saito-Kurokawa conjecture, now a theorem due to Maass, Andrianov and Zagier, gives a one-to-one correspondence between classical modular forms and the *Maass Spezialschar*, a certain subspace of the Siegel modular forms of genus $g = 2$. This bijection is realized by an identity between the $L$-functions of the eigenforms in both spaces. Further work and generalizations of Saito-Kurokawa have been studied by many, notably Duke-Imamoglu [7], Breulmann-Kuss [4], Kohnen [11], Ikeda [10] and Miyawaki [12].

A classical Hecke eigenform is the $\Delta$-function, defined by

$$\Delta(z) := \frac{1}{1728}(E_4(z)^3 - E_6(z)^2) = \sum_{n \geq 1} \tau(n) q^n.$$

*Congruences* associated to the Fourier coefficents $\tau(n)$ have also been famously studied. For example, the congruence

$$\tau(n) \equiv n^{11} + 1 \pmod{691}$$

(meaning $\tau(n) - n^{11} - 1$ is always divisible by 691) dates back to Ramanujan, while the more recent and very deep work of Deligne and Serre, and Swinnerton-Dyer (see, e.g., [6], [13], and [15]) involving Galois representations finally "explains" this congruence, as well as the other (provably finite set!) of congruences associated to $\Delta(z)$. During a colloquium talk in Bonn in 2003, Harder provided a remarkable conjecture generalizing the $\tau(n)$ congruence above to the Siegel modular setting. A detailed account of Harder's work can be found in the book under review, as well as supporting evidence due to van der Geer and Faber.

**Conjecture** (Harder). *Let $f$ be an eigenform of weight $r$ on $\mathrm{SL}_2(\mathbb{Z})$. Then under suitable hypotheses, there exists a Siegel modular form $F$ that is an eigenform of the Hecke algebra with eigenvalue $\lambda(p)$ for $T(p)$ such that for suitable primes $\ell$ and integers $j, k$, the following congruence holds for all primes $p$:*

$$\lambda(p) \equiv p^{k-2} + a(p) + p^{j+k-1} \pmod{\ell}.$$

The authors of *The 1-2-3 of Modular Forms* succeed in providing a tangible yet thorough account of the theory of modular forms in various guises and applications, discussing in more detail the topics mentioned here as well as many more. Undoubtedly, due to both its breadth and readability, this book will be a useful source for mathematicians, both novice and expert, wishing to read more about modular forms and a beautiful theory.

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