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Limit cycles of differential equations, by Colin Christopher and Chengzhi Li, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2007, viii+171 pp., ISBN 978-3-7643-8409-8

It is well known that very simple nonlinear (autonomous, first-order) ordinary differential equations can represent quite complicated dynamic behaviour. The most famous example is perhaps the Lorenz equation in three dimensions, in which all terms are linear except for two quadratic terms. If we however restrict to two-dimensional equations

$$\dot{x} = \frac{dx}{dt} = P(x,y), \qquad \dot{y} = \frac{dy}{dt} = Q(x,y), \tag{1}$$

with P and Q polynomial and relatively prime, then the situation is substantially simpler, at least with respect to the richness of the asymptotic behaviour of the orbits. The Poincaré-Bendixson theorem implies that a bounded ω-limit set (and hence also a bounded α-limit set) of an orbit has to be either a singularity (also called a zero, an equilibrium, a critical point, or a stationary point), a periodic orbit, or a graphic, i.e., a connected set consisting of a finite number of singularities and a finite number of regular orbits that are homoclinic or heteroclinic between these singularities. Unbounded ω- and α-limit sets can have a slightly more complicated structure but are still very simple. To get a fairly good idea of the phase portrait of a differential system (P,Q) as in (1) it suffices to draw the *separatrix skeleton* and *extended separatrix skeleton*, as precisely defined in [DLA]. The first essential ingredient of these skeletons is the set of singularities whose cardinality is bounded by mn (Bézout's theorem), where m and n are the respective degrees of P and Q (we can suppose that $m \leq n$). Besides the singularities the separatrix skeleton also contains a finite number of *separatrices* emanating from the singularities. These separatrices can be found by desingularization (also called blow-up) of the singularities. The computer program P4 is able to draw such separatrix skeletons. See [DLA] for an introduction to the use of P4 as well as to a detailed overview of the methods and theoretical results on which it is based.

In extending the separatrix skeleton, the most difficult task consists of adding the periodic orbits. In 2002 a nice survey paper [I2] on this subject appeared in the *Bulletin of the American Mathematical Society* by the hand of Ilyashenko. I have to recall some information from that paper in order to precisely situate the contents of the book in review. I will however limit this to a minimum and will concentrate on supplementary results and more recent developments. I also do not want to go into detail on the subjects that are presented in the book.

Periodic orbits in polynomial planar differential systems can be isolated or belong to an annulus of periodic orbits. The former are called *limit cycles*. If all orbits in the neighbourhood have the limit cycle C as ω-limit set (resp., α-limit set) we call C stable or attracting (resp., unstable or repelling). If all orbits on one side have C as ω-limit set and all orbits on the other side have C as α-limit set we call C semi-stable.

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If the periodic orbit is not isolated, we say that we have a center in which C belongs to a *period annulus*. The boundary of the period annulus consists of a (connected) inner boundary and an outer boundary. If the outer boundary is bounded, then it is also connected.

It can all look simple, but finding the periodic orbits, limit cycles as well as centers, is not simple at all. For an individual system, to check that a (regular) point belongs to a period annulus or a limit cycle is in general a nontrivial question. It is, e.g., not known how many limit cycles a system (P, Q) can have. Dulac, in a paper of 1923 [Dul] claimed to prove that the number of limit cycles of an individual system is finite. His method relies on desingularization of the singularities and compactification of the plane. His proof contains a flaw that was detected in the mid-1970s. Filling that gap proved to be a highly nontrivial task that was accomplished independently by Ilyashenko in [I1] and Écalle in [E]. As observed by Smale in [Sm], these two papers have yet to be thoroughly digested by the mathematical community. The situation is quite clear, when, after compactification and desingularization, only hyperbolic singularities show up. It gets much more complicated when semi-hyperbolic singularities also enter into the game.

Although the usual Poincaré compactification and homogeneous blow-up can be applied in this general process, it is good to mention that, for the study of specific situations as well as in more general questions, it is often better to use less homogeneous constructions such as *Poincaré-Lyapunov compactification* and *quasi-homogeneous blow-up*. A didactic introduction to both can be found in [DLA].

The solution of what was known as *Dulac's problem* does not however answer the original question that Hilbert posed in 1900 as the second part of his 16th problem (see [H]). We state it as follows: *What is the maximum number $H(n)$ of limit cycles for system* (1) *with degrees of P and Q at most n, and what are the possible relative configurations of these limit cycles?* Concerning the configurations, a general result was obtained in [LR] (see also [DLA]). Let us focus on the first question. $H(n)$ is now called the Hilbert number of the family of polynomial systems (P, Q) with $\deg P = m$ and $\deg Q = n$ and $m \leq n$. The solution of Dulac's problem does not imply the existence of a finite Hilbert number $H(n)$, also called *the finiteness question in Hilbert's* 16*th problem.* Of course $H(1) = 0$, but the finiteness of $H(n)$, with $n \geq 2$, is still an open problem. From [S] we know that $H(2) \geq 4$. From [DF] we know that a stratification of the space of quadratic planar vector fields, in strata with a given number of limit cycles, is not algebraic nor even analytic.

A method for proving the finiteness of $H(n)$ has been proposed by Roussarie (see [R1]). It is based on compactification of both the phase plane and the parameter space, reducing the problem to proving the *finite cyclicity* of *limit periodic sets.* In [DRR] this program was started for the case $n = 2$ by listing 121 limit periodic sets whose cyclicity has to be investigated. Since then many of these cases have been treated.

Smale also showed interest in Hilbert's 16th problem when making his list of problems for the 21st century [Sm]. His 13th problem addresses the same question as Hilbert's does with emphasis on polynomial Liénard equations.

The latter come from second-order scalar differential equations $\ddot{x}+f(x)\dot{x}+g(x) = 0$ and can be represented in a so-called phase plane as $\dot{x} = y$, $\dot{y} = -g(x)-f(x)y$ and in a so-called Liénard plane as $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$, where $F(x) = \int_0^x f(s)ds$. Taking f of degree m and g of degree n, we can denote the related Hilbert number as

$H(m,n)$. Liénard equations with $m=1$ are called *classical Liénard equations*, and Smale's 13th problem essentially concerns these. Known results are $H(1,1)=0$, $H(1,2)=1$, $H(2,1)=1$, $H(2,2)=1$, and $H(3,1)=1$ (see, e.g., [DL]). For all other $H(m,n)$ even finiteness is not known. By compactification of both the phase plane and the Liénard family it was proven in [R2], for n even, and in [CD], for n odd, that the finiteness question can be reduced to a study of singular perturbation problems (perturbing from systems with a curve of singularities). More information on compactification and desingularization of spaces of Liénard equations can be found in [D]. The singular perturbation problems correspond to Liénard equations with very large coefficients and are now under systematic investigation. This study already led (see [DPR]) to a counterexample of a conjecture in [LMP] on the number of limit cycles of classical Liénard equations.

Writing the Liénard system as $\dot{x}=y-(x^N+a_{N-1}x^{N-1}+\cdots+a_1x), \dot{y}=-x$, we know from [CD] that, when N is even, then under a uniform bound on the $|a_j|$ the number of limit cycles is uniformly bounded and, more specifically, there can be at most $N/2$ *large amplitude limit cycles*. The method of proof is in the line of Roussarie's program. With a completely different approach [IP] provides, in case N is odd and $|a_j|\leq C$, the explicit but extremely large upper bound $\exp(\exp(C^{14^N}))$.

It may be interesting to mention that Smale in his 13th problem proposes proving that the growth of $H(n)$ is polynomial. In any event, for N either even or odd, the related singular perturbation problems remain to be studied.

Another perturbation problem that has a much longer tradition than the singular one is perturbation from a period annulus. In perturbing from a period annulus, limit cycles can be created from the boundary or from the interior. The latter is a purely analytic question; the former is quite more irregular, dealing with limit behaviour of analytic functions, and it is, therefore, often not considered at all in many papers. An exception has to be made for an inner boundary consisting of a nondegenerate singularity, which is also an analytic problem. In any case all limit cycles under consideration coincide with fixed points of the Poincaré return map, which is an analytic function near any periodic orbit in the period annulus (at least with respect to an analytic transverse section). There also exists an analytic integrating factor f, changing a system (P^o,Q^o), along the interior of the period annulus, to a Hamiltonian system (fP^o,fQ^o). The perturbation problem can hence be seen as a perturbation from an analytic Hamiltonian system with Hamiltonian H. If we write the system, using 1-forms, as the equivalent Pfaffian equation $dH+\varepsilon\omega=0$, then in first-order approximation the limit cycles are given by the isolated zeros of the integral

$$I(h)=\int_{H^{-1}(h)}\omega. \tag{2}$$

Of course this integral depends on the same parameters that the equation is subject to. The study of the zero set of the integral is hence an important question. A great deal of attention has already been given to the case in which both H and ω are polynomial. The integral is then called an *Abelian integral* and the search for the maximum number $Z(m,n)$ of isolated zeros of such integrals, depending only on the degrees m and n of H and ω, respectively, is called the infinitesimal (or weak or tangential) Hilbert's 16th problem. Varchenko and Khovanskii have proven that, for given m and n, the number $Z(m,n)$ is finite. From Petrov-Khovanskii we know that the asymptotic growth of $Z(m,n)$ with respect to n is linear, for a fixed Hamiltonian. We refer to [I2] for more detailed information. However, except for a very limited number of cases, the precise value of $Z(m,n)$ is not known. It is known for H of degree ≤ 3. Also $\widetilde{Z}(n) = Z(n+1,n)$ is known only for $n = 2$. Recently the upper bound $\widetilde{Z}(n) \leq 2^{2^{P(n)}}$, for some explicit polynomial $P(n)$, has been provided in [BNY]; it is certainly far from sharp. In current research much attention also goes to integrals (2) for which H or ω is no longer a polynomial, but where some generalizations are admitted, such as taking H of Darboux type. More information can be found in the book that we are reviewing.

It is necessary to observe that merely studying the integrals (2), even when they are quite generic, is not sufficient for obtaining all limit cycles that can be perturbed from the boundary of a period annulus. This has first been shown in [DR] for a 2-saddle cycle as boundary, giving rise to so-called “alien” limit cycles.

Before presenting the contents of the book, let me mention that the recent book [RS] provides a didactic and elementary introduction to the center-focus problem, with a great deal of attention to the practical implementation of the methods.

Contents of the book. The book itself consists of two separate sets of course notes presented separately by Colin Christopher and Chengzhi Li.

Part 1. *Around the Center-Focus Problem* (C. Christopher)

This part considers some of the topics concerning the (local) center-focus problem and the (global) integrability of polynomial systems.

When system (1) has a singularity whose linearization gives a center (i.e., it has nonzero imaginary eigenvalues), then how can one determine whether the system has at that point a center or a (weak) focus ? This problem is far from being trivial and is closely related with the search for integrability conditions.

Chapter 1 deals with the center-focus problem and the bifurcation of limit cycles from the center. Chapters 2 and 3 discuss Darboux integrability and Liouvillian integrability, respectively. Chapter 4 is devoted to symmetric centers with applications to Liénard equations, and Chapter 5 deals with Cherkas's systems. Monodromy is introduced in Chapter 6 and applied to Abel equations. The tangential center-focus problem, a local version of the formerly mentioned tangential (or infinitesimal) Hilbert's 16th problem, is introduced in Chapter 7. Chapter 8 deals with the monodromy of hyperelliptic Abelian integrals, i.e., integrals (2) with $H(x,y) = y^2 + f(x)$, for some polynomial f. Lotka-Volterra systems are considered in Chapter 9, and a short introduction to other approaches is presented in Chapter 10. Part 1 can hence be considered to be an interesting and quite detailed introduction to the seemingly most algebraic aspects in Hilbert's 16th problem.

Part 2. *Abelian Integrals and Applications to the weak Hilbert's* 16*th problem* (C. Li)

This part is devoted to the introduction of some basic concepts and methods in the study of the infinitesimal Hilbert's 16th problem. Chapters 1 and 2 essentially deal with the relation between the study of Abelian integrals and the study of limit cycles. Attention is also given to higher-order approximations of the Poincaré return map (or a related displacement function) and to the period function.

In Chapter 3 different methods are used to study the zeros of the elliptic integrals associated with the Bogdanov-Takens (local) bifurcation.

In Chapter 4 a unified proof is presented for the different cases that have to be treated in proving that $\widetilde{Z}(2) = 2$. These notes constitute a nice and detailed introduction to the basics of the theory.

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