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SELECTED MATHEMATICAL REVIEWS

MR0441965 (56 #356) 14C30; 14F15

Deligne, Pierre

Théorie de Hodge. I. (French)

Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 425–430.
Gauthier-Villars, *Paris*, 1971.

In this article the author develops his philosophy concerning cohomology theories for algebraic schemes. Motivated by the theory of motives, he sketches a way to compare l-adic cohomology and classical cohomology. For a scheme X over a field K which is a finitely generated extension of the prime field, $\mathrm{Gal}(\overline{K}/K)$ acts on the l-adic cohomology groups $H^q(X, \mathbf{Z}_l)$, where l is a prime different from the characteristic of K. Let A be a ring of finite type over $\mathbf{Z}$ with quotient field K. A closed point x of $\mathrm{Spec}(A)$ gives rise to a conjugacy class of Frobenius elements φ_x in $\mathrm{Gal}(K_{\mathrm{nr}}/K)$. A Galois module H is called pure of weight n if there exists an A such that, if α is an eigenvalue of ${\varphi_x}^{-1}$ on H and q_x is the number of elements in the residue field k_x, then α is an algebraic integer all of whose conjugates have absolute value ${q_x}^{n/2}$. According to the Weil conjectures, proved by the author [Inst. Hautes Études Sci. Publ. Math. No. 43 (1974), 273–307; MR0340258 (49 #5013)], if X is smooth and projective over K, then $H^i(X, \mathbf{Z}_l)$ is pure of weight i. The main principle is that every Galois module H coming from l-adic cohomology should admit a unique weight filtration W such that ${\mathrm{Gr}_n}^W(H)$ is pure of weight n for all n. A mixed Hodge structure consists of data (H, W, F), where H is a finitely generated abelian group, W is an increasing filtration of $H \otimes Q$ and F is a decreasing filtration of $H \otimes \mathbf{C}$ such that F induces for each n a decomposition ${\mathrm{Gr}_n}^W(H \otimes \mathbf{C}) = \oplus_{p+q=n} H^{pq}$ with $H^{\overline{pq}} = H^{qp}$. By classical Hodge theory, if X is a projective (or Kähler) manifold, then $H^i(X, \mathbf{Z})$ carries a Hodge structure which is pure of weight i, i.e., with $W_j = 0$ for $j < i$, $W_j = H^i(X, \mathbf{Z})$ for $j \geq i$. An analogy between Galois modules and mixed Hodge structures is indicated, the weight filtration on Galois modules being given by eigenvalues of Frobenius elements. By the main principle and this analogy it is expected that for every scheme X of finite type over $\mathbf{C}$, $H^i(X, \mathbf{Z})$ carries a natural mixed Hodge structure. This has been proved by the author for smooth schemes [Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57] and for arbitrary schemes [ibid. No. 44 (1974), 5–77].

This analogy is applied to the comparison of a projective scheme X over a Henselian discrete valuation ring with smooth generic fiber on one side and a projective family $f\colon X \to D$, where D is the unit disc and f is smooth over $D \backslash \{0\}$, on the other side. As a result, it is announced that for every tangent vector u of D at 0 a mixed Hodge structure may be constructed on the cohomology of the generic fiber of f, whose weight filtration is completely determined by the monodromy of the family.

{For the entire collection see MR0411874 (54 #3).}

From MathSciNet, June 2009

J. H. M. Steenbrink

MR0498551 (58 #16653a) 14C30; 14F15

Deligne, Pierre

Théorie de Hodge. II. (French)

Institut des Hautes Études Scientifiques. Publications Mathématiques, *no.* 40 (1971), 5–57.

MR0498552 (58 #16653b) 14C30; 14F15

Deligne, Pierre

Théorie de Hodge. III. (French)

Institut des Hautes Études Scientifiques. Publications Mathématiques, *no.* 44 (1974), 5–77.

Part II contains the construction of a mixed Hodge structure on $H^*(U)$, where U is a smooth algebraic variety over $\mathbf{C}$. The construction is algebraic, but one needs classical Hodge theory to show that it works. The main idea is the introduction of a weight filtration W on $H^*(U)$ by means of an embedding $j\colon U \to X$ of U in a smooth complete variety X such that $X \smallsetminus U = Y$ is a divisor with normal crossings on X. The weight filtration is associated with the Leray spectral sequence. The Hodge filtration part of the mixed Hodge structure comes from a filtration F on the logarithmic de Rham complex $\Omega^{\cdot}_X(\log Y)$ given by $F^p(\Omega^k_X(\log Y)) = \Omega^k_X(\log Y)$ if $k \geq p$, $F^p(\Omega^k_X(\log Y)) = 0$ if $k < p$, and from the canonical isomorphism $H^*(U, \mathbf{C}) \simeq \mathbf{H}^*(X, \Omega^{\cdot}_X(\log Y))$ where $\mathbf{H}$ denotes hypercohomology. The author develops the necessary tools from homological algebra, among which there is a study of the several filtered spectral sequences, associated with a complex with more than one filtration. This is applied to the category of mixed Hodge structures. It is shown that every morphism of mixed Hodge structures is strictly compatible with Hodge and weight filtrations. The paper ends with some applications: a proof of the global invariant cycle theorem, a result concerning the semisimplicity of certain categories of families of mixed Hodge structures and a theorem about homomorphisms of abelian schemes.

The main result of part III, a fundamental paper, is the construction of a mixed Hodge structure on the cohomology of any algebraic variety X over $\mathbf{C}$. In this first step, X is replaced by a simplicial scheme $X.$ which is the complement of a divisor D with normal crossings in a simplicial scheme $\overline{X}.$, such that all $\overline{X}_n$ are smooth complete varieties and such that the morphism $X. \to X$ is of cohomological descent. The second step is an application of the results of part II to all X_n, and a glueing using techniques from homological algebra. The resulting mixed Hodge structure on $H^*(X)$ is independent of the choices made and is functorial in X. If X is smooth [complete], the Hodge numbers h^{pq} of $H^n(X)$ can only be nonzero if $p + q \geq n$ [$p + q \leq n$]. Rational homology manifolds behave like smooth varieties in this respect. Among the examples one finds the computation of the cohomology of classifying spaces and of smooth projective hypersurfaces. The paper also contains a treatment of the connection between mixed Hodge structures of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ and 1-motives.

{Part I has been reviewed [Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 425–430, Gauthier-Villars, Paris, 1971; MR0441965 (56 #356)].}

From MathSciNet, June 2009

J. H. M. Steenbrink

MR0340258 (49 #5013) 14G13

Deligne, Pierre

La conjecture de Weil. I. (French)

Institut des Hautes Études Scientifiques. Publications Mathématiques, *no.* 43 (1974), 273–307.

This is without question the most important paper in algebraic geometry to have appeared in the last ten years (since H. Hironaka's proof of resolution of singularities in characteristic zero [Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326; MR0199184 (33 #7333)]). Deligne has proved the Riemann hypothesis for varieties over finite fields! The superficial reader may be dismayed that the proof uses the full strength of the Artin-Grothendieck étale cohomology theory as it is set down in the 2544 pages of SGA4 and SGA7 [*Théorie des topos et cohomologie étale des schémas, Tome 1: Théorie des topos* (Sém. de Géométrie Algébrique du Bois-Marie 1963–1964, SGA4), Lecture Notes in Math., Vol. 269, Springer, Berlin, 1972; *Théorie des topos et cohomologie étale des schémas, Tome 2* (Sém. de Géométrie Algébrique du Bois-Marie 1963–1964, SGA4), Lecture Notes in Math., Vol. 270, Springer, Berlin, 1972; *Théorie des topos et cohomologie étale des schémas, Tome 3* (Sém. de Géométrie Algébrique du Bois-Marie 1963–1964, SGA4), Lecture Notes in Math., Vol. 305, Springer, Berlin, 1973; *Groupes de monodromie en géométrie algébrique, I* (Sém. de Géométrie Algébrique du Bois-Marie 1967–1969, SGA7I), Lecture Notes in Math., Vol. 288, Springer, Berlin, 1972; *Groupes de monodromie en géométrie algèbrique, II* (Sém. de Géométrie Algébrique du Bois-Marie 1967–1969, SGA7II), Lecture Notes in Math., Vol. 340, Springer, Berlin, 1973] and the various loose exposés (available in preprint form only) of the still unpublished SGA5. In fact the author does an excellent job of isolating and stating clearly precisely which results he is using, thus providing the reader with a valuable skeleton key to those weighty tomes. Because the actual proof is quite readable, and has already been summarized by J.-P. Serre ("Valeurs propres des endomorphismes de Frobenius (d'après P. Deligne)", to appear in Séminaire Bourbaki, 1973/1974, 26$^{\text{ème}}$ année), I will only discuss the sources of some of the ideas that go into the proof.

Recall that for any variety X over a finite field $\mathbf{F}_q$, its zeta function $Z(X/\mathbf{F}_q, T)$ is defined as the formal power series $\exp(\sum_{n\geq 1} N_n T^n/n)$, where N_n is the number of points of X with coordinates in the field $\mathbf{F}_{q^n}$. Thus the zeta-function of X provides a sort of Diophantine summary of X.

In 1949, A. Weil [Bull. Amer. Math. Soc. **55** (1949), 497–508; MR0029393 (10,592e)] made his famous conjectures about the zeta-function of a projective, non-singular n-dimensional variety X over $\mathbf{F}_q$ (generalizing what he himself had proved for X a curve, an abelian variety or a Fermat hypersurface).

(1) $Z(X/\mathbf{F}_q, T)$ is a rational function of T.

(2) Moreover,

$$Z(X/\mathbf{F}_q, T) = P_1(T)P_3(T)\cdots P_{2n-1}(T)/P_0(T)P_2(T)\cdots P_{2n}(T), \tag{1}$$

where $P_i(T) = \prod_{j=1}^{b_i}(1-\alpha_{ij}T)$, $|\alpha_{ij}| = q^{i/2}$, the last equality being the "Riemann hypothesis" for varieties over finite fields.

(3) Under $\alpha \mapsto q^n/\alpha$, the $\alpha_{i,j}$ are carried bijectively to the $\alpha_{2n-i,j}$. This is a functional equation for $T \mapsto 1/q^nT$.

(4) In case X is the "reduction modulo p" of a nonsingular projective variety $\mathbf{X}$ in characteristic zero, then b_i is the ith topological Betti number of $\mathbf{X}$ as a complex manifold.

The moral is that the topology of the complex points of $\mathbf{X}$, expressed through the classical cohomology groups $H^i(\mathbf{X}, \mathbf{C})$, determines the form of the zeta-function of X, i.e., determines the Diophantine shape of X. Weil gave a heuristic argument for this, as follows [Proceedings of the International Congress of Mathematicians (Amsterdam, 1954), Vol. III, pp. 550–558, Noordhoff, Groningen, 1956; MR0092196 (19,1078a)]. Among all elements of the algebraic closure of $\mathbf{F}_p$, the elements of $\mathbf{F}_q$ are singled out as the fixed points of the Frobenius morphism $x \mapsto x^q$. More generally, if $\mathbf{x} = (\cdots, x_i, \cdots)$ is a solution of some equations which are defined over $\mathbf{F}_q$, then $F(\mathbf{x}) \stackrel{\text{dfn}}{=} (, \cdots, {x_i}^q, \cdots)$ will also be a solution of the same equations, and the point $\mathbf{x}$ will have its coordinates in $\mathbf{F}_q$ precisely when $F(\mathbf{x}) = \mathbf{x}$. Thus F is an endomorphism of our variety X over $\mathbf{F}_q$, and $N_n = \#\mathrm{Fix}(F^n)$; thus $\mathrm{Z}(X/\mathbf{F}_q, T) = \exp(\sum (T^n/n)\#\mathrm{Fix}(F^n))$.

Suppose that we consider instead a compact complex manifold $\mathbf{X}$, and an endomorphism $\mathbf{F}$ of $\mathbf{X}$ with reasonable fixed points. Then the Lefschetz fixed point formula would give us $\#\mathrm{Fix}(\mathbf{F}^n) = \sum (-1)^i \mathrm{trace}(\mathbf{F}^n | H^i(\mathbf{X}, \mathbf{C}))$, which is formally equivalent to the identity

$$\text{(2)} \qquad \exp\Big(\sum_{n \geq 1} (T^n/n)\#\mathrm{Fix}(\mathbf{F}^n)\Big) = \prod_{i=0}^{2n} \det(1 - T\mathbf{F} |\, H^i(\mathbf{X}, \mathbf{C}))^{(-1)^{i+1}}.$$

The search for a "cohomology theory for varieties over finite fields" which could justify this heuristic argument has been responsible, directly and indirectly, for much of the tremendous progress made in algebraic geometry during the past twenty-five years. Weil's proofs of the Riemann hypothesis for curves over finite fields had already necessitated his *Foundations of algebraic geometry* [Amer. Math. Soc., New York, 1946; MR0023093 (9,303c); revised edition, Providence, R.I., 1962; MR0144898 (26 #2439)]. Around the same time, O. Zariski had also begun emphasizing the need for an abstract algebraic geometry. His disenchantment with the lack of rigor in the Italian school had come after writing his famous monograph *Algebraic surfaces* [Springer, Berlin, 1935; Zbl **10**, 377] which gave the "state of the art" as of 1934. The possibility of transposing to abstract algebraic varieties with their "Zariski topology" the far-reaching topological and sheaf-theoretic methods that had been developed by Picard, Lefchetz, Hodge, Kodaira, Leray, Cartan, ... in dealing with complex varities was implicit in Weil's lecture notes "Fibre spaces in algebraic geometry" [mimeographed lecture notes, Math. Dept., Univ. of Chicago, Chicago, Ill., 1952 (1955)]. This transposition was carried out by Serre in his famous article FAC [Ann. of Math. (2) **61** (1955), 197–278; MR0068874 (16,953c)]. From the point of view of the Weil conjectures, however, this theory was still inadequate, for when applied to varieties in characteristic p it gave cohomology groups that were vector spaces in characteristic p, so could only give "mod p" trace formulas, i.e., could only give "mod p" congruences for numbers of rational points.

(I) **l-adic cohomology**: After some false starts (e.g., Serre's Witt vector cohomology [International Symposium on Algebraic Topology, pp. 24–53, Univ. Nac. Autónom. de México, Mexico City, 1958; MR0098097 (20 #4559); Amer. J. Math. **80** (1958), 715–739; MR0098100 (20 #4562)]) and B. M. Dwork's "unscheduled" (because apparently non-cohomological) proof [ibid. **82** (1960), 631–648;

MR0140494 (25 #3914)] of the rationality conjecture (1) for any-variety over $\mathbf{F}_q$, M. Artin and A. Grothendieck developed a "good" cohomology theory based on the notion of étale covering space, and generalizing Weil's l-adic matrices [see the third, fourth and fifth references to SGA4 above]. In fact, they developed a whole slew of theories, one for each prime number $l \neq p$, whose coefficient field was the field $\mathbf{Q}_l$ of l-adic numbers. Each theory gave a factorization of the zeta-function $\mathrm{Z}(T) = \prod_{i=0}^{2n} P_{i,l}(T)^{(-1)^{i+1}}$ into an alternating product of $\mathbf{Q}_l$-adic polynomials, satisfying conjecture (3). In the case when $\mathbf{X}$ could be lifted to $\mathbf{X}$ in characteristic zero, they proved that $P_{i,l}$ was a polynomial of degree $b_i(\mathbf{X})$. However, they did not prove that the $P_{i,l}$ in fact had coefficients in $\mathbf{Q}$, nor a fortiori that the $P_{i,l}$ were independent of l. This meant that in the factorization of an individual $P_{i,l}$, $P_{i,l}(T) = \prod_{j=1}^{b_i}(1 - \alpha_{i,j,l}T)$, the roots $\alpha_{i,j,l}$ were only algebraic over $\mathbf{Q}_l$, but possibly not algebraic over $\mathbf{Q}$, and so they might not even have archimedean absolute values. (Of course, by a theorem of Fatou, the actual reciprocal zeros and poles of the rational function $\mathrm{Z}(T)$ are algebraic integers; the problem is that there may be cancellation between the various $P_{i,l}$ in the l-adic factorization of the zeta-functions.)

So the question became one of how to introduce archimedean considerations into the l-adic theory. Even before the l-adic theory had been developed, Serre [Ann. of Math. (2) **7** (1960), 392–394; MR0112163 (22 #3018); correction, MR **22**, p. 2545], following a suggestion of Weil [see the tenth reference above, p. 556], had formulated and proved a Kählerian analogue of the Weil conjectures, making essential use of the Hodge index theorem. In part inspired by this, in part by his own earlier (1958) realization that the Castelnuovo inequality used by Weil was a consequence of the Hodge index theorem on a surface, Grothendieck in the early sixties formulated some very difficult positivity and existence conjectures about algebraic cycles, the so-called "standard conjectures" [cf. S. Kleiman, *Dix exposés sur la cohomologie des schémas*, pp. 359–386, North-Holland, Amsterdam, 1968; MR0292838 (45 #1920)], whose truth would imply the independence of l and the Riemann hypothesis.

Much to everyone's surprise, the author managed to avoid these conjectures altogether, except to deduce one of them from the Weil conjectures, the "hard" Lefschetz theorem giving the existence of the "primitive decomposition" of the cohomology of a projective non-singular variety, a result previously known only over $\mathbf{C}$, and there by Hodge's theory of harmonic integrals. The rest of the "standard conjectures" remain open. In fact, the generally accepted dogma that the Riemann hypothesis could not be proved before these conjectures had been proved [cf., J. Dieudonné, *Cours de géométrie algébrique, Vol. I: Aperçu historique sur le development de la géométrie algébrique*, especially p. 224, Presses Univ. France, Paris, 1974; *Vol. II: Précis de géométrie algébrique élémentaire*, 1974] probably had the effect of delaying for a few years the proof of the Riemann hypothesis.

(II) **The new ingredients**: So what was it that finally allowed the Riemann hypothesis for varieties over finite fields to be proved? There were two principal ingredients. (1) Monodromy of Lefschetz pencils: In the great work of S. Lefschetz [*L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, 1924; reprinting, 1950; MR0033557 (11,456c)] on the topology of algebraic varieties, he introduced the technique of systematically "fibering" a projective variety by its hyperplane sections, and then expressing the cohomology of that variety in terms of the cohomology of those fibers. The general Lefschetz theory was successfully transposed into l-adic cohomology, but it didn't really bear Diophantine fruit until D. A. Každan and G. A. Margulis proved that the "monodromy group" of a Lefschetz pencil of odd fibre dimension was as "large as possible". The author realized that if the same result were true in even fibre dimension as well, then it would be possible to inductively prove the independence of l and the rationality of the $P_{i,l}$ of X, by recovering them as generalized "greatest common divisors" of the $P_{i,l}$ of the hyperplane sections. But the Každan-Margulis proof was Lie-algebra theoretic in nature, via the logarithms of the various Picard-Lefschetz transformations in the monodromy group. The restriction to odd fibre dimension was necessary because in that case the Picard-Lefschetz transformations were unipotent, thus had interesting logarithms, while in even fibre dimension they were of finite order. Soon thereafter, N. A'Campo [Invent. Math. **20** (1973), 147–169; MR0338436 (49 #3201)], found a counterexample to a conjecture of Brieskorn that the local monodromy of isolated singularities should always be of finite order. Turning sorrow to joy, Deligne realized that A'Campo's example could be used to construct (non-Lefschetz) pencils which would have unipotent local monodromy. These he used to make the Každan-Margulis proof work in even fibre-dimension as well, and so to establish the "independence of l" and rationality of the $P_{i,l}$ [cf. J.-L. Verdier, Séminaire Bourbaki, $25^{\text{ème}}$ année (1972/1973), Exp. No. 423, pp. 98–115. Lecture Notes in Math., Vol. 383, Springer, Berlin, 1974].

With this result, the importance of monodromy considerations for Diophantine questions was firmly established. (2) Modular forms, Rankin's method, and the cohomological theory of L-series: In the years after the Weil conjectures were first formulated, experts in the theory of modular forms began to suspect a strong relation between the Weil conjectures and the Ramanujan conjecture on the order of magnitude of $\tau(n)$. Recall that the $\tau(n)$ are the q-expansion coefficients of the unique cusp form Δ of weight twelve on $\mathrm{SL}_2(\mathbf{Z})$: $\Delta(q) = q(\prod_{n\geq 1}(1-q^n))^{24} = \sum \tau(n)\cdot q^n$. As an arithmetic function, $\tau(n)$ occurs essentially as the error term in the formula for the number of representations of n as a sum of 24 squares. The Ramanujan conjecture is that $|\tau(n)| \leq n^{11/2}d(n)$, $d(n) = \#$ (divisors of n). According to Hecke theory (which had been "prediscovered" by Mordell for Δ), the Dirichlet series corresponding to Δ admits an Euler product: $\sum_{n\geq 1}\tau(n)\cdot n^{-s} = \prod_p (1/1-\tau(p)\cdot p^{-s}+p^{11-2s})$.

The truth of the Ramanujan conjecture for all $\tau(n)$ is then a formal consequence of its truth for all $\tau(p)$ with p prime: $|\tau(p)| \leq 2p^{11/2}$. This last inequality may be interpreted as follows. Consider the polynomial $1-\tau(p)T+p^{11}T^2$ and factor it: $1-\tau(p)T+p^{11}T^2 = (1-\alpha(p)T)(1-\beta(p)T)$. Then the Ramanujan conjecture for $\tau(p)$ is equivalent to the equality $|\alpha(p)| = |\beta(p)| = p^{11/2}$. If there were a projective smooth variety X over $\mathbf{F}_p$ such that the polynomial $1-\tau(p)\cdot T+p^{11}T^2$ divided $P_{11}(X/\mathbf{F}_p, T)$, then the Riemann hypothesis for X would imply the Ramanujan conjecture for $\tau(p)$. The search for this X was carried out by Eichler, Shimura, Kuga, and Ihara [cf. Y. Ihara, Ann. of Math. (2) **85** (1967), 267–295; MR0207655 (34 #7470); M. Kuga and G. Shimura, ibid. (2) **82** (1965), 478–539; MR0184942 (32 #2413)]. They constructed an X which "should have worked", but because their X was not compact and had no obvious smooth compactification, its polynomial P_{11} did not necessarily have all its roots of the correct absolute value. The author then showed how to compactify their X and how to see that the Hecke polynomial $1-\tau(p)T+p^{11}T^2$ divided a certain factor of P_{11}, the roots of which factor would have the "correct" absolute values if the Weil conjectures were true. Thus the truth of the Ramanujan conjecture became a consequence of the universal truth of the Riemann hypothesis for varieties over finite fields.

In 1939 R. A. Rankin [Proc. Cambridge Philos. Soc. **35** (1939), 351–372; MR0000411 (1,69d); correction, MR **1**, p. 400] had obtained the then-best estimate for $\tau(n)$ (namely $\tau(n) = O(n^{29/5})$) by studying the poles of the Dirichlet series $\sum(\tau(n))^2 \cdot n^{-s}$. R. P. Langlands [*Lectures in modern analysis and applications, III*, pp. 18–61, Lecture Notes in Math., Vol. 170, Springer, Berlin, 1970; MR0302614 (46 #1758)] pointed out that the idea of Rankin's proof could easily be used to prove the Ramanujan conjecture, provided one knew enough about the location of the poles of an infinite collection of Dirichlet series formed from Δ by forming even tensor powers: for each even integer $2n$ one needed to know the poles of the function represented by the Euler product $\prod_p \prod_{i=0}^{2n} (1/(1-\alpha(p)^i \beta(p)^{2n-i} p^{-s}))^{\binom{2n}{i}}$.

The author studied Rankin's original paper in an effort to understand the remarks of Langlands. He realized that for L-series over curves over finite fields (instead of L-series over $\mathrm{Spec}(\mathbf{Z})$), Grothendieck's cohomological theory [A. Grothendieck, Séminaire Bourbaki, Vol. 1964/1965, Exp. No. 279, facsimile reproduction, Benjamin, New York, 1966; see MR(33 #54201)] of such L-series together with the Každan-Margulis monodromy result gave an a priori hold on the poles: Rankin's methods could therefore be combined with Lefschetz pencil-monodromy techniques to yield the Riemann hypothesis for varieties over finite fields, and with it the Ramanujan-Petersson conjecture as a corollary.

(III) **Other Applications**: Another arithmetic application is the estimation of exponential sums in several variables. Though technically difficult, the idea goes back to Weil [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207; MR0027006 (10,234e)], who showed how the Riemann hypothesis for curves over finite fields gave the "good" estimate for exponential sums in one variable.

As for geometric applications, we have already mentioned the hard Lefschetz theorem which is promised for the sequel to the present paper. There is also a whole chain of ideas built around the "yoga of weights", Grothendieck's catchphrase for deducing results on the cohomology of arbitrary varieties by assuming the Riemann hypothesis for projective non-singular varieties over finite fields. The whole of the author's "mixed Hodge theory" for complex varieties [Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57; "Théorie de Hodge, III", to appear in Inst. Hautes Études Sci. Publ. Math.], developed before his proof of the Riemann hypothesis, is intended to prove results about the cohomology of these varieties which follow from the Riemann hypothesis and from the systematic application of Hironaka's resolution of singularities. The recent work of the author, Griffiths, Morgan and Sullivan on the rational homotopy type of complex varieties is also considerably clarified by the use of the Riemann hypothesis.

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Nicholas M. Katz

MR0601520 (83c:14017) 14G13; 10H10

Deligne, Pierre

La conjecture de Weil. II. (French)

Institut des Hautes Études Scientifiques. Publications Mathématiques, *no.* 52 (1980), 137–252.

This paper is the sequel to an earlier one by the author [same journal No. 43 (1974), 273–307; MR0340258 (49 #5013)]. By an essentially new method, which he calls the method of Hadamard-de la Vallée-Poussin because it involves showing that a large class of L-functions do not vanish on the line $\mathrm{Re}\, s = 1$, the author generalizes his results on the absolute values of the eigenvalues of Frobenius acting on étale cohomology to the case of twisted coefficients. He applies his results to prove the hard Lefschetz theorem and the local invariant cycle theorem, as well as to the study of the homotopy type of an algebraic variety.

Let X_0 be an algebraic variety defined over a finite field $\mathbf{F}_q$, and let X be the base extension of X_0 to the algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$. (The convention of dropping a subscript 0 to indicate base extension to $\mathbf{F}$ will be used without comment throughout this review.) Let x be a geometric point of X. The fundamental group $\pi_1(X_0, x)$ is an extension of $\hat{\mathbf{Z}}$ by $\pi_1(X, x)$. A smooth l-adic sheaf on X_0 [resp. Weil sheaf on X_0] is given by a continuous representation of $\pi_1(X_0, x)$ [resp. of $W(X_0, x) =$ subgroup of $\pi_1(X_0, x)$ mapping to $\mathbf{Z} \subset \hat{\mathbf{Z}}$] on a finite-dimensional vector space V over a finite extension of $\mathbf{Q}_l$.

Given a closed point $y \in X_0$, there is a conjugacy class $[F_y] \subset \pi_1(X_0, x)$ associated to the inverse of the Frobenius in $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q(y))$. A Weil sheaf E_0 on X_0 is i-pure of weight n for a given isomorphism i of $\overline{\mathbf{Q}}_l$ onto $\mathbf{C}$ if for all $y \in X_0$ the eigenvalues of F_y all have absolute value $q^{n/2}$. E_0 is pure of weight n if it is i-pure of weight n for any i. For example, if $f_0\colon Y_0 \to X_0$ is smooth and projective, $R^m f_* \mathbf{Q}_l$ is pure of weight m by the Weil conjectures. E_0 is mixed if it is an iterated extension of pure sheaves.

For E_0 on X_0 a Weil sheaf, the cohomology $H_c^*(X, E)$ inherits a $\mathbf{Z}$-action and hence a notion of i-weights (computed for the inverse of the canonical generator of $\mathbf{Z}$). The main result in the paper under review is that E_0 i-mixed of weights $\leq n$ implies that $H_c^r(X, E)$ has i-weights $\leq n + r$.

To see the power of this result, suppose X_0 is an open smooth curve and that E_0 and G_0 are pure of weights n and m with $n \leq m$. Using the duality between compactly supported and ordinary cohomology, $H^1(X, \mathrm{Hom}(E, G))$ is seen to have weights ≥ 1. In particular there are no Frobenius invariants, hence no extensions of E_0 by G_0 nonsplit over X. This semisimplicity result, applied with $X_0 \subset \mathbf{P}^1$ parametrizing smooth members of a Lefschetz pencil on a variety V_0, and E the sheaf of middle dimensional cohomology groups on the fibres of the pencil, yields $E = E^\pi \oplus W$ with $\pi = \pi_1(X)$. W has no π invariants or coinvariants so E and W are perpendicular under the intersection pairing. This is equivalent to the classical assertion that any invariant vanishing cycle is trivial, and the hard Lefschetz theorem follows.

Let $Z(E_0, t) = \prod_y \det(I - F_y t | E_y)^{-1}$. The Grothendieck cohomological formula gives $Z(E_0, t)$ as a product $\det(I - Ft | H_c^i(X, E)^{(-1)^i})$. The proof of the main theorem is reduced to the case when E_0 has i-weight 0 and X_0 is an open curve. By duality it suffices to show the weights of $H_c^i(X, E) \leq 1$. An elementary argument based on the convergence of the infinite product for $Z(E_0,t)$ shows that these weights ≤ 2.

One assumes inductively the weights $\leq 1+2^{-k}$, and one considers $E_0 \boxtimes E_0$ on $X_0 \times X_0$. If $Y_0 \subset X_0 \times X_0$ is a hyperplane section one shows that the weights on $H^1(Y, E_0 \boxtimes E_0|Y)$ are integral and strictly less than 2. (It is here that the Hadamard-de la Vallée-Poussin method is used.) Fibering $X_0 \times X_0$ by a Lefschetz pencil, the above is sufficient to show the weights on $H^2(X\times X, E\boxtimes E) \leq 2+2^{-k}$, which gives $1+2^{-k-1} \geq$ weights of $H^1(X,E)$.

Let ω_s be the character $q^{-\deg(x)\cdot s}$ on $W = W(X_0,\overline{x})$, where $s \in \mathbf{C}$ and $\deg\colon W \to \mathbf{Z}$. The Hadamard-de la Vallée-Poussin idea is based on considering L-functions $L(\tau\omega_s)$ where X_0 is a curve and τ is a unitary representation of W. Let $\nu(\tau)$ be the residue at $s=1$ of

$$\frac{-L'}{L}(\tau\omega_s) = \sum_{n,x} \log N(x)\cdot \mathrm{Tr}(\tau(F_x^n))\mathrm{N}(x)^{-ns}.$$

One knows that $\nu(\tau)$ is defined, $\nu(1)=1$, $\nu(\tau)=\nu(\overline{\tau})$, and $\nu(\tau)\leq 0$ for $\tau \neq 1$. Extending ν to the Grothendieck group of virtual unitary representations by additivity and observing the terms on the right above are positive for s real and $\mathrm{tr}(\tau)>0$, one also has $\nu(\rho\otimes\overline{\rho}) \geq 0$ for any virtual unitary representation ρ. The author proves a general lemma valid for any group W to the effect that such a function ν on the category of virtual unitary representations necessarily satisfies $\nu(\tau)=0$ for τ irreducible unitary except $\tau=1$ and possibly one other τ defined by a character of order 2. In the case at hand, such an exotic τ would correspond to a curve (double cover of X_0) whose zeta function had no pole at $s=1$, and this cannot occur.

By a curious misprint, the running head throughout the paper is "la conjoncture de Weil". More appropriate might have been "la conjunction de Weil et Deligne".

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