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MINKOWSKI IN KÖNIGSBERG 1884: A TALK IN LINDEMANN'S COLLOQUIUM

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Introduction

The mathematical-physico seminar at the university of Königsberg in the 1880s shaped the early scientific career of both David Hilbert and Hermann Minkowski. The cover of this issue of the *Bulletin of the American Mathematical Society* displays a page from the *Protokollbuch* of the colloquium at Königsberg, written in Minkowski's own hand on the day of his talk in May 1884. That a seminar even existed at this time and in this place is a tribute to the reforms in the German educational system earlier in the century. This paper briefly traces those developments and their influence on scientific education in general to provide a frame for understanding Minkowski and his mathematical training. This broader context allows a more meaningful survey of Minkowski's talk where he established an estimate for the minimum of a positive definite quadratic form depending only on its discriminant and degree and interpreted this result geometrically. This geometric approach reveals the first glimpse of Minkowski's later work in the geometry of numbers.

1. The mathematical-physico seminar at the Albertus-University in Königsberg

Neohumanistic Bildungsreform. At the beginning of the nineteenth century, after the political collapse of the old state in 1806, the structure of Prussia's government underwent significant changes. Within this development, a far-reaching new approach towards education and teaching originated, called *neohumanistic Bildungsreform*, mainly due to the conceptual efforts of Wilhelm von Humboldt

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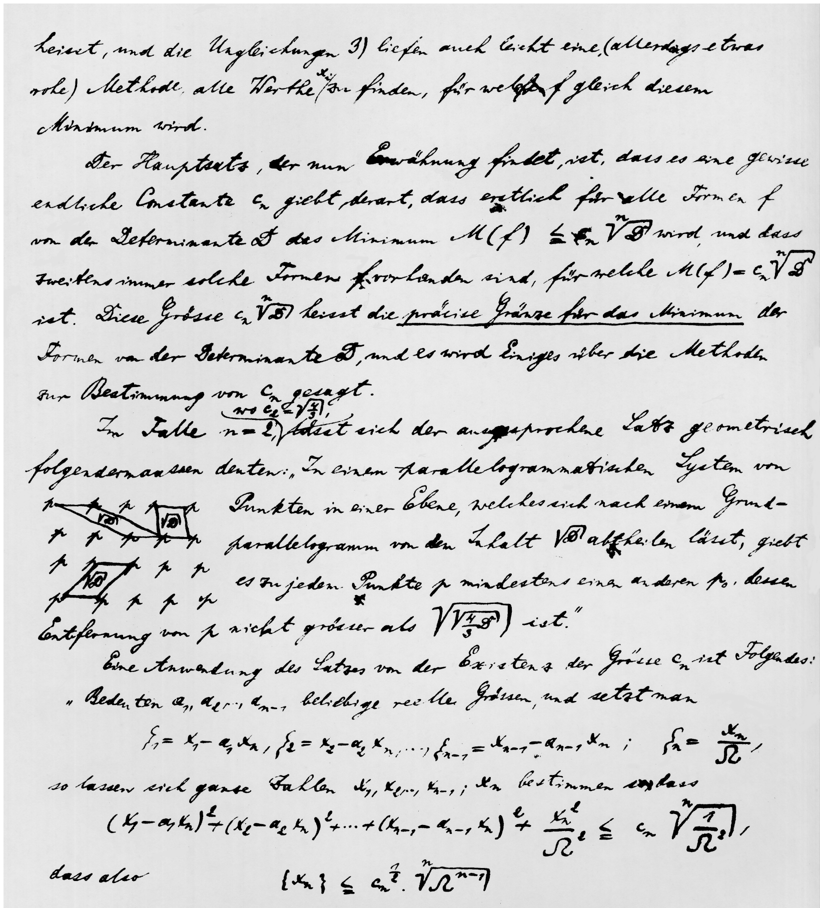
heisst, und die Ungleichungen 3) liefern auch leicht eine (allerdings etwas rohe) Methode, alle Werthe x_i zu finden, für welche f gleich diesem Minimum wird.

Der Hauptsatz, der nun Erwähnung findet, ist, dass es eine gewisse endliche Constante c_n giebt, derart, dass erstlich für alle Formen f von der Determinante $\mathfrak{D}$ das Minimum $M(f) \leq c_n \sqrt[n]{\mathfrak{D}}$ wird, und dass zweitens immer solche Formen vorhanden sind, für welche $M(f) = c_n \sqrt[n]{\mathfrak{D}}$ ist. Diese Grösse $c_n \sqrt[n]{\mathfrak{D}}$ heisst die präcise Gränze für das Minimum der Formen von der Determinante $\mathfrak{D}$, und es wird Einiges über die Methoden zur Bestimmung von c_n gesagt.

Im Falle $n=2$, (wo $c_2 = \sqrt{\tfrac{4}{3}}$), lässt sich der ausgesprochene Satz geometrisch folgendermaassen deuten: „In einem parallelogrammatischen System von Punkten in einer Ebene, welches sich nach einem Grundparallelogramm von dem Inhalt $\sqrt{\mathfrak{D}}$ abtheilen lässt, giebt es zu jedem Punkte p mindestens einen anderen p_0, dessen Entfernung von p nicht grösser als $\sqrt{\sqrt{\tfrac{4}{3}\mathfrak{D}}}$ ist."

Eine Anwendung des Satzes von der Existenz der Grösse c_n ist Folgendes: „Bedeuten $a_1, a_2, \ldots, a_{n-1}$ beliebige reelle Grössen, und setzt man

$$\xi_1 = x_1 - a_1 x_n,\ \xi_2 = x_2 - a_2 x_n, \ldots, \xi_{n-1} = x_{n-1} - a_{n-1} x_n;\quad \xi_n = \frac{x_n}{\Omega},$$

so lassen sich ganze Zahlen $x_1, x_2, \ldots, x_{n-1}; x_n$ bestimmen, dass

$$(x_1 - a_1 x_n)^2 + (x_2 - a_2 x_n)^2 + \cdots + (x_{n-1} - a_{n-1} x_n)^2 + \frac{x_n^2}{\Omega^2} \leq c_n \sqrt[n]{\frac{1}{\Omega^2}},$$

dass also

$$\{x_n\} \leq c_n^{\frac{1}{2}} \cdot \sqrt[n]{\Omega^{n-1}}$$

About the cover. From the exposé written by Minkowski in the *Protokollbuch* of Lindemann's colloquium, Albertus University Königsberg. (Courtesy of Foundation Otto Volk, Institute of Mathematics, University of Würzburg, Germany.)

and Freiherr von Stein. They introduced a notable distinction between "Gymnasium" and university. The former institution aimed at an integral general education (*ganzheitlicher allgemeiner Bildung*) as a basis for an individual unfolding of the own self. The reform emphasized that elements of the natural sciences and historical knowledge should complement the previous canon of elements of knowledge based on languages and grammar. This change required highly qualified teachers at the Gymnasium. As a result, aside from the faculties of theology, medicine, and law, the philosophical faculty, which included mathematics, the sciences, philosophy, philology, and history, gained a new role by introducing coherent schemes of courses which were to assure this intended level of training. The concept of intertwining research and teaching formed another essential feature of this reform.

Partly enforced by the transfer of governmental actions to Königsberg, the government gave special attention and support to the Albertus-University in Königsberg, all in the name of the *neohumanistic Bildungsreform.* Based on these govermental fundings and directions, the university evolved during the nineteenth century into a well respected place both for learning science and pursuing new research in science. This strong scientific record at Albertus-University needed more than governmental funds to strengthen and grow—it needed the work and reputation of eminent scientists, researchers, and teachers alike, in particular, in the natural sciences.

These developments in Königsberg had a significant impact on the restructuring of the Prussian educational system as a whole and the structure of the curricula in the individual disciplines as well.

The mathematical-physico seminar. The mathematical-physico seminar at the university of Königsberg emerged as a site for teaching and learning *science*, in particular, new science. The structure of this seminar grew out of—and furthered—this larger educational process.

In 1834 the physicist Franz E. Neumann (1798–1895) together with the mathematician Carl Gustav Jacobi (1804–51) established the mathematical-physico seminar at the university. They were supported substantially by Friedrich Wilhelm Bessel (1784–1846), professor of astronomy since 1810 and director of the observatory.

The seminar was composed of the division for pure and applied mathematics (mechanics, physical astronomy) and the division for mathematical physics. It was the first official seminar in Prussia to incorporate mathematical methods into physics instruction, a step with far-reaching consequences. Ultimately, this institution became the center of a school of mathematical physics, shaping the formation of this discipline in a substantial way. Jacobi directed the mathematical division from 1834 to 1844. His student Friedrich Julius Richelot (1808–75) succeeded him and oversaw the division until 1875. Heinrich Weber, born in 1842 and a former student in Königsberg from 1863–65, held an ordinary professorship at the Polytechnic in Zürich when he accepted a call to the Albertus-University as the successor to Richelot in 1875. At that time Woldemar Voigt, the last doctoral student of Franz Neumann, had already defended his thesis (1874) and had moved to Leipzig where he worked as a secondary school teacher. Concurrently, he completed his *Habilitation* at the university of Leipzig, then he accepted a position at Königsberg as extraordinary professor of theoretical physics. It didn't take long until he took over Neumann's lectures in theoretical physics. His role in pursuing the work and program of physics instruction of his former teacher is discussed in [8, p. 436]. In August 1883, Voigt assumed a position in Göttingen as ordinary professor for theoretical physics and director of the mathematical-physico institute.

The joint efforts of Neumann, Jacobi, and Bessel towards new forms of educational training for teachers led to essential changes in the studies of mathematics and physics at the university. Previously, students could pursue their studies rather freely, even without the formal constraints of a curriculum. Students in the new seminar had to take a well structured, hierarchically ordered plan of studies. The students admitted to the seminar had to study and master the contents of various required lectures in mathematics and physics. For students who could not handle the mathematical sophistication required for the seminar, basic courses were offered as prerequisites. More advanced students could get aquainted with questions and results of a more research-related nature in personal discussions with their professors in the seminar. Working through problem sets or dealing with more expanded topical themes were essential elements of the students' work in the seminar.

Despite a small budget and a very limited number of rooms available for the seminar, not to mention an insufficient library, this new orientation of the mathematical training and the instruction in physics strengthened scientific research. The aim that students should also acquire scientific competence in their studies gained considerable momentum.

The different implications of the mathematical-physico seminar can be characterized as follows.

First, upon completion of their studies, i.e., after passing the final examination, most of the students worked as secondary school teachers (Gymnasium) where their higher level of knowledge and training had a direct impact on their teaching. This new emphasis conveyed a large amount of inspiration. Minkowski's teacher, Dr. Louis Hübner, serves as a prominent example. Born in 1850, he studied mathematics in Königsberg, in particular with F. Richelot. In 1873 he received a prize for a manuscript which was later accepted as his dissertation at Marienwerder. Then, in 1876, he moved to the Altstädtische Gymnasium where he worked as a teacher in the *prima*. At this place special attention was given to the content of the mathematics courses, even beyond the official curriculum. An analysis of the problems given in the oral and written examinations for the *Abitur* reveals that Hübner edged the students beyond the expected mathematical curriculum.

Second, young scientists transferred the basic educational ideas inherent in the seminar to other universities where they were adapted and further developed. Examples are given by Gustav Robert Kirchhoff (1824–87) in Heidelberg [3, vol. 1, Section 12] and Rudolph Alfred Clebsch (1833–72) in Giessen [8, pp. 424–426]. Thus, the mathematical-physico seminar at the university in Königsberg played a pivotal role in the reform of Prussian universities [3], [8].

Third, within the scientific context of mathematics and physics, the seminar strongly influenced the development of research areas and methodology.

Fourth, due to the pedagogical commitment and work of Franz Neumann for over four decades until 1875, the seminar has to be viewed as the beginning of the new discipline to be called mathematical or theoretical physics.

2. Minkowski as a student

Studies in Königsberg and Berlin. Hermann Minkowski was born on June 22, 1864, in Alexotas, Russia, now Lithuania. A difficult financial situation in 1872 led the family to move to Prussia where they settled in Königsberg and where Minkowski attended the Altstädtische Gymnasium. There he received a solid mathematical education based on the Prussian curricula at the time, which stressed classical languages, German and French, mathematics, physics, history, and geography. His mathematics teacher in his last years Dr. Louis Hübner, a former student at the Albertus-University and a member of its mathematical-physico seminar, taught his course on quite a high level of mathematical insight [12]. In April 1880, Minkowski graduated and enrolled at the Albertus-University in Königsberg where he studied mathematics. His principal teachers were H. Weber and W. Voigt. Weber was very familiar with B. Riemann's work and ideas, he was the author of the *Theorie der algebraischen Funktionen einer Veränderlichen*, and he worked in algebra, number theory, and the theory of functions. His mathematical expertise and his open scholarly instruction shaped Minkowski's development considerably. However, in 1883, Weber moved to the recently founded Technical University in Berlin-Charlottenburg, and, after another year, to Marburg. After Frobenius had declined a chair at Königsberg, Ferdinand Lindemann (1852–1939) accepted the call as the successor of Weber, starting October 1, 1883. Soon thereafter, in the spring of 1884, the young Adolf Hurwitz (1859–1919) worked as an extraordinary professor at his side. Lindemann directed the mathematical division of the seminar, while Paul Volkmann (1856–1938) oversaw the physical division. As a member of the seminar for nine semesters, Volkmann had the perfect training for this position, and he could continue the tradition begun by Neumann. In the meantime he worked as an assistent of Voigt and he became *Privatdozent* in 1882 [8, p. 441].

This was the institutional setting when Minkowski returned to Königsberg in the spring of 1884 from a three semester leave he had spent at the university of Berlin. There he had pursued his work in the theory of reduction for positive definite quadratic forms, one of the themes in Minkowski's previous research. His unpublished draft of a manuscript, entitled *Zur Theorie der Reduction der wesentlich positiven quadratischen Formen*, was written in November 1883. Therein, he discussed a new notion of a reduced form, different from Hermite's approach, to tackle the general problem of reduction for n-ary forms. Lindemann and Hurwitz played central but different roles in Minkowski's scientific development. One of Minkowski's fellow students was David Hilbert. Their friendship had already begun three years earlier but at this time it expanded to include Hurwitz as a mentor and friend.

The students of mathematics had to attend certain required lectures. These courses covered the foundations and more advanced material in analysis, the theory of functions, algebra, number theory, geometry, and mathematical physics as well as astronomy. The lectures, actually given in the summer term of 1884, were the following:

- Algebraische Analysis, 4 hrs
- Differentialrechung, 4 hrs
- Theorie der Determinanten (in the seminar), 1 hr
- Analytische Geometrie der Ebene, 4 hrs
- Synthetische Geometrie, 4 hrs
- Ueber Moebius' barycentrischen Calcul, 2 hrs
- Theorie der elliptischen Funktionen, 4 hrs
- Functionen mit mehreren complexen Variablen, 1 hr
- Theorie der linearen Differentialgleichungen (in the seminar),
- Elasticitätstheorie, 4 hrs
- Ueber die Fortpflanzung der Electricität in Drähten (in the seminar)
- Ueber einige Beobachtungsmethoden, 1 hr
- Ueber physikalische Begriffe und absolute Masse, 2 hrs
- Geodäsie, 4 hrs
- Sphärische Astronomie, 2 hrs.

The situation, however, was not very comfortable for either professors or students. There were no rooms that had been specifically designed for the needs of lectures in mathematics. Next to the lecturing desk (*Katheder*), one found only two blackboards posted on an easel. Lindemann's request to provide a large blackboard made out of slate at the wall with a podium in front of it was first turned down by the dean but eventually realized [4, p. 89].

Lindemann's colloquium. Even more important was the weekly mathematical colloquium that Lindemann had established in 1884. It offered a forum for talks and scientific exchange at the research level. Lindemann writes in his memoirs:

> I found a number of older students who were even better versed than me and prepared in an excellent way by my predecessor Weber. Pursuing the habit as initiated by Clebsch and Klein, these older students gave me the chance to establish a colloquium for the purpose of which I assembled these students together with Hurwitz once per week in my flat. As a rule, one of the participants gave a longer lecture possibly on their own investigations, and moreover the recent literature was discussed. [4, p. 89, my translation]

Lindemann's *Protokollbuch* records the details of the weekly events. The topics included results of the participants and surveys of the work of others in mathematics and mathematical physics. Its first session took place on May 6, 1884, offering a lecture by Lindemann in which he talked on a generalization of methods of Mittag-Leffler and Weierstrass to represent analytic functions. On May 13, 1884, David Hilbert discussed, under the title *Realität der Wurzeln von Gleichungen* 5*.ten Grades*, some results published by Sylvester in Crelle's Journal, volume 87, 1879. In the next session of the colloquium on May 20, 1884, Minkowski gave a lecture on positive quadratic forms. As in the previous sessions the audience consisted of Lindemann, Hurwitz, Paul Volkmann, Emil Wiechert (1861–1928), Ernst Meyer, Hilbert, C. Mey, and Orlowsky. The entry in the *Protokollbuch* for this day lists a short review by Wiechert on work by Warburg and Hartwig in the *Festschrift d. naturf. Gesellschaft*. At the same time, the topic *Apolaritaet*, a talk to be given by Mey, was announced for the next session. Its summary then appears as a report on work by Lindemann on geometric representations of covariants of binary forms, published in 1877, and some subsequent developments. The themes dealt with in the following months ranged from class number relations (Hurwitz), over Fechner-Weber's law (covered by Wiechert) to the optical work of W. Voigt presented by his pupil Volkmann.

3. Minkowski's talk on positive quadratic forms, May 20, 1884, Königsberg

Following the entry written by Minkowski in the *Protokollbuch* of the colloquium at Königsberg, he started his lecture on May 20, 1884, with the general problem of finding an approximate solution in the integers for a system of given equations. In particular, he had the case of linear homogenous equations

$$\xi_i = \sum_k a_{ik} x_k = 0 \; (i = 1, \dots, m)$$

with real coefficients in mind. He explained that there is no loss in generality in assuming that the number of equations equals the number of indeterminates. Of course, the trivial solution was not of interest, so one had to look for a solution such that the "equation" holds as an approximation. Minkowski pointed out that results in probability theory as well as geometric and analytic considerations had led him to analyze the term

$$\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$$

because this expression measures the distance of a solution to the trivial solution $\xi_1 = 0, \xi_2 = 0, \ldots, \xi_n = 0$. Writing the term in the variables x_i, $i = 1, \ldots, n$, it represents a positive definite quadratic form

$$f = \sum c_{ik} x_i x_k$$

of determinant $D > 0$. Now the basic problem was to solve the equations $\xi_i = 0$ approximately in the integers so that f is as small as possible.

Given a bound G, Minkowski continued, there are only finitely many integral solutions x not equal to zero so that the value of f taken on x is smaller than the bound. As a consequence, he stated, one can find in a finite number of steps the smallest positive value M which can be represented by f. By definition, the value M will be called the minimum of the positive form f.

The following paragraph of the entry captures the essence of Minkowski's talk. He asserted that there is a constant c_n such that for a positive form f in n variables with determinant D, one has the estimate

$$M(f) \leq c_n \sqrt[n]{D}.$$

Moreover, he pointed out, there are always forms for which one has equality in this estimate. The term $c_n \sqrt[n]{D}$ was going to be called the *präzise Gränze für das Minimum*, the precise bound for the minimum of positive forms with determinant D. Without giving further detail, Minkowski stated in the entry that he discussed various methods to determine c_n.

In the case $n = 2$, where $c_2 = \sqrt{\frac{4}{3}}$, Minkowski indicated the geometric significance of this precise bound: it is a result about the existence of a shortest vector in the lattice attached to f. As illustrated by a small drawing in the entry, one has to view this assertion in light of the geometric interpretation of positive forms as first suggested by C. F. Gauss. In his review [1] of the 1831 treatise *Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen* by Ludwig August Seeber, Gauss had indicated the following construction: Starting with a positive quadratic form $q = ax^2 + 2bxy + cy^2$, the equation $\cos\alpha = b/\sqrt{ac}$ uniquely determines an acute or obtuse angle α so that one can attach a system of coordinates to f whose axes form the angle α. With regard to this system, an arbitrary point has the coordinates $x\sqrt{a}$, $y\sqrt{c}$, and the form q gives the square of the distance of this point to the origin. So far as the variables refer to integers, the form q makes reference to a system of points which is given as the intersection of two families of equidistant parallel lines. In this way the plane is subdivided into parallelograms whose vertices just form this system of points. By adapting this geometric representation of q, Minkowski denoted these points in his drawing by the small letter p, and he also indicated that the determinant of q coincides with the square of the area of each parallelogram.

Minkowski concluded his talk with an application of the estimate in Diophantine approximation. Given arbitrary real numbers $a_1, \ldots, a_{n-1}$, one can find integers $x_1, \ldots, x_n$ so that the ratios $\frac{x_i}{x_n}$, all with the same denominator, approximate these numbers with an error term of the order x_n^{-k} where $k = \frac{n}{n-1}$.

Applications of this type had already appeared in Hermite's work relating a problem in number theory to the arithmetic theory of quadratic forms as laid out in his 1850 letters to Jacobi [2]. In fact, applications of this type led Hermite to develop a reduction theory for forms in n variables in these letters. There, as one of the main results, he proved an estimate for the ratio of the minimum of a definite quadratic form in n variables to the nth root of its determinant by a constant depending only on n. Thus, the result stated in Minkowski's talk was not new at that time, but the question of determining the precise bound for the minimum of the form was a major unresolved issue. Minkowski was fully aware of Hermite's reduction theory. In November 1883, while still in Berlin, Minkowski had written a draft of a manuscript, titled *Zur Theorie der Reduction der wesentlich positiven quadratischen Formen*, where he realized that the notion of a reduced form in the sense of Hermite was no longer adequate in dealing with the question regarding the precise bound and, hence, required change. Minkowski's considerations in this unpublished manuscript make it clear how Minkowski was led later on to introduce new defining conditions for a given form to be reduced.[1] Geometrical concepts had not directed his methods of investigation in this manuscript of 1883. However, a few months later, in his talk given in the colloquium at Königsberg, Minkowski emphasized the geometric point of view in the search for the precise bound for the minimum of a quadratic form. The geometric assertion still seems to be a reinterpretation of a result obtained otherwise, but it is this approach which unfolded in the following years as an essential ingredient in Minkowski's work.

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[1] The content of the manuscript *Zur Theorie der Reduction der wesentlich positiven quadratischen Formen* and the remarkable shift in this approach to reduction theory are discussed in [11].