

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by

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MR0387496 (52 #8338) 22E40; 20G10

Borel, Armand

Stable real cohomology of arithmetic groups.

Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975).

We begin with a quote from the author’s introduction: “Let Γ be an arithmetic subgroup of a semisimple group G defined over the field of rational numbers \mathbf{Q} . The real cohomology $H^*(\Gamma)$ of Γ may be identified with the cohomology of the complex Ω_X^Γ of Γ -invariant smooth differential forms on the symmetric space X of maximal compact subgroups of the group $G(\mathbf{R})$ of real points of G . Let I_G be the space of differential forms on X which are invariant under the identity component $G(\mathbf{R})^0$ of $G(\mathbf{R})$. It is well known to consist of closed (in fact harmonic) forms, whence a natural homomorphism $j^*: I_G \rightarrow H^*(\Gamma)$. Our main result (7.5) gives a range computable from the algebraic group structure of G , up to which j^* is an isomorphism. For a given group, it is rather small, about $\frac{1}{4}n$ for $\mathbf{SL}_n(\mathbf{Z})$ for instance; however, we may consider a sequence (G_n, Γ_n) , where G_n and Γ_n are as G and Γ , for which our range tends to infinity, and of injective morphisms $f_n: (G_n, \Gamma_n) \rightarrow (G_{n+1}, \Gamma_{n+1})$. Since much is known about I_G , this allows us to determine the ‘stable real cohomology’ $H^q(\Gamma_n)$, n large, and $H^*(\varinjlim \Gamma_n)$ for sequences of classical groups (Section 11). For example, $H^*(\varinjlim \mathbf{SL}_n(\mathbf{Z})) \dots$ is an exterior \dots algebra over infinitely many generators, one for each dimension of the form $\dots 4i + 1$ ($i = 1, 2, \dots$). The result for $\mathbf{SL}(\mathbf{Z})$ then implies that the dimension $K_i(\mathbf{Z}) \otimes \mathbf{Q}$ is one if $i \equiv 1 \pmod{4}$ and zero otherwise ($i \geq 2$).”

More generally, the author computes $\dim(K_i \mathfrak{o} \otimes \mathbf{Q})$ ($i \geq 2$) when \mathfrak{o} is an order in a central division algebra D over an algebraic number field k . Indeed, if r_1 [r_2] is the number of real [complex] places of k , then for $i \geq 2$, one has that $\dim(K_i \mathfrak{o} \otimes \mathbf{Q})$ is periodic of period four, and equals 0, $r_1 + r_2$, 0, r_2 , depending on whether $i = 0, 1, 2, 3 \pmod{4}$, respectively. (Also, applications to L -theory are given.) These results suggest a connection between the groups $K_i \mathfrak{o}$ and ζ_k , the Dedekind ζ -function of k (at least in the case when \mathfrak{o} is the ring of integers in k). Thus the above result implies that if $i \geq 1$ is an integer then the order of the zero of ζ_k at $-i$ is equal to $\dim(K_{2i+1} \mathfrak{o} \otimes \mathbf{Q})$. This result is empirical, but S. Lichtenbaum [*Algebraic K-theory, II: “Classical” algebraic K-theory and connections with arithmetic* (Proc. Conf., Seattle Res. Center, Battelle Memorial Inst., 1972), pp. 489–501, Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973] has formulated some striking conjectures concerning the possible relationship between K -groups and ζ -functions. Subsequent work by the present author, Lichtenbaum, and Coates has shed further light on this; in particular, see the author’s subsequent article [Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. I, pp. 435–442, Canad. Math. Congr. Montreal, Que., 1975].

We quote further from the author’s introduction: “The homomorphism j^* is defined for any discrete subgroup Γ of $G(\mathbf{R})$. If $G(\mathbf{R})/\Gamma$ is compact, then, as a

simple consequence of Hodge theory, j^* is injective in all dimensions. . . . Results of Y. Matsushima [Osaka Math. J. **14** (1962), 1–20; MR0141138 (25 #4549)] show the existence of a constant $m(G(\mathbf{R}))$, determined by the Lie algebra of $G(\mathbf{R})$, up to which j^* is surjective. H. Garland had already shown [Ann. of Math. (2) **94** (1971), 534–548; MR0297733 (45 #6785)] that if $G(\mathbf{R})/\Gamma$ is only assumed to have finite invariant volume, and if every cohomology class in a given dimension $q \leq m(G(\mathbf{R}))$ is representable by a square integrable form, then Matsushima’s arguments carry over without change and show that j^* is surjective in dimension q . Moreover, Garland and W. C. Hsiang [Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 354–360; MR0228504 (37 #4084)] gave, for Γ arithmetic, a range up to which this condition is fulfilled (a so-called ‘square integrability criterion’). Our main concern is therefore the injectivity of j^* in the arithmetic case, but our discussion will also include the surjectivity and the square integrability criterion.”

The author’s proof rests on a geometric construction due to himself and J.-P. Serre [see #8337 above]. It suffices to assume Γ is torsion free. The construction of the author and Serre allows one to view X/Γ as the interior of a compact manifold with corners, \overline{X}/Γ . Also, assuming only that $G(\mathbf{R})/\Gamma$ has finite invariant volume, the author remarks that one can show that, “given an integer $m'(G)$, j^* is bijective at least up to $\min(m(G(\mathbf{R})), m'(G))$ if there exists a subcomplex C of Ω_X^Γ such that up to $m'(G)$: (i) $C \rightarrow \Omega_X^\Gamma$ induces an isomorphism in cohomology; (ii) $I_C \subset C$; (iii) C^q consists of square integrable forms”.

To prove the existence of the subcomplex C , the author uses the compactification \overline{X}/Γ mentioned earlier where, in fact, C is defined by means of certain growth conditions near the boundary of \overline{X}/Γ . These growth conditions will guarantee that (ii) is satisfied in all dimensions and that (iii) is satisfied up to a certain constant defined in terms of the \mathbf{Q} -roots of \mathbf{G} . As for (i), one introduces a presheaf \mathcal{F} on \overline{X}/Γ . “For y on the boundary and U a neighborhood of y in \overline{X}/Γ , $\mathcal{F}(U)$ is the space of forms on $U \cap (X/\Gamma)$ satisfying the growth condition which is part of the definition of C . For $y \in X/\Gamma$, $\mathcal{F}(U)$ is simply the space of differential forms defined on U . This presheaf is a sheaf whose space of sections on \overline{X}/Γ and X/Γ are C and Ω_X^Γ , respectively. Since the inclusion $X/\Gamma \hookrightarrow \overline{X}/\Gamma$ is a homotopy equivalence, it suffices to prove that \mathcal{F} is a fine resolution of \mathbf{R} in \overline{X}/Γ . The ‘finesses’ is easy, so that the main point is to check a local Poincaré lemma near a boundary point, i.e., on a Siegel set. This is done by showing that the usual homotopy operator used to prove the Poincaré lemma on Euclidean space does not alter our growth condition.”

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Howard Garland

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Borel, Armand

Stable real cohomology of arithmetic groups. II.

Manifolds and Lie groups (Notre Dame, Ind., 1980), pp. 21–55, *Progr. Math.*, 14, Birkhauser, Boston, Mass., 1981.

This paper is a sequel to one with the same title (but without the roman numeral) by the author [Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975); MR0387496 (52 #8338)]. We refer the reader to the review of that article (call it

SI) for background and here paraphrase part of the author's introduction to this sequel (call it SII).

Let G be a Zariski-connected isotropic reductive \mathbf{Q} -group, Γ an arithmetic subgroup of $G(\mathbf{R})$, and X the symmetric space for $G(\mathbf{R})$. Let I^q denote the space of Γ -invariant and G -invariant harmonic q -forms on X . In SI, a constant $d(G)$ was given such that the natural map $j^q: I^q \rightarrow H^q(\Gamma, \mathbf{C})$ is bijective for all $q \leq d(G)$. The three main steps of the proof were (i) the cohomology of Γ can be computed from a complex of differential forms satisfying a certain logarithmic growth condition at infinity; (ii) below a certain dimension $c(G)$, these forms are all square-integrable; (iii) below a certain dimension $m(G)$, j^q is injective and has as image the space of cohomology classes representable by square-integrable forms.

In SII, the case of nontrivial coefficients defined by a finite-dimensional complex representation E of G is also considered. Other growth conditions are also investigated, and it is proved that $H^*(\Gamma, E)$ can be computed using a complex of forms of moderate growth or a complex of weakly λ -bounded forms (where λ is a dominant linear form on the Lie algebra of a maximal \mathbf{Q} -split torus). The proof is similar to its analogue in SI. These forms are square-integrable up to a constant $C(G, \lambda, E)$, which constant is studied in great detail in SII. Vanishing theorems for relative Lie algebra cohomology with coefficients in infinite-dimensional unitary representations occurring in the spectrum of Γ , rather than the arguments originally due to Matsushima in SI, are used to improve the constant $m(G)$ and to provide an analogue $M(G, E)$ for nontrivial E . This leads to the bijectivity of j^q below a certain dimension and also to the vanishing of $H^q(\Gamma, E)$ below a certain dimension when E has no nontrivial subrepresentation.

Two related topics are treated in SII. (A) One can compute the cohomology with compact supports of $\Gamma \backslash X$ with a complex of fast decreasing forms or with a complex of weakly λ -bounded forms for $\lambda < 0$. It follows that a nonzero fast decreasing Γ -invariant harmonic form is not cohomologous to zero in $H^*(\Gamma, E)$ and is cohomologous to a closed form with compact support. This applies in particular to harmonic cusp forms. (B) The main theorem about bijectivity of j^q and vanishing of $H^q(\Gamma, E)$ remains valid for S -arithmetic groups and groups of rational points, with essentially the same bounds. The proof uses Bruhat-Tits buildings. As a result, the stability theorems of SI extend to S -arithmetic groups and groups of rational points, and their consequences for K -theory and L -theory are also valid for rings of S -integers and number fields.

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