BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 2, April 2010, Pages 363–365
S 0273-0979(10)01300-5
Article electronically published on February 2, 2010

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
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MR0387496 (52 #8338) 22E40; 20G10

Borel, Armand

Stable real cohomology of arithmetic groups.

Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975).

We begin with a quote from the author's introduction: "Let Γ be an arithmetic subgroup of a semisimple group G defined over the field of rational numbers $\mathbf{Q}$. The real cohomology $H^*(\Gamma)$ of Γ may be identified with the cohomology of the complex ${\Omega_X}^\Gamma$ of Γ-invariant smooth differential forms on the symmetric space X of maximal compact subgroups of the group $G(\mathbf{R})$ of real points of G. Let I_G be the space of differential forms on X which are invariant under the identity component $G(\mathbf{R})^0$ of $G(\mathbf{R})$. It is well known to consist of closed (in fact harmonic) forms, whence a natural homomorphism $j^*\colon I_G \to H^*(\Gamma)$. Our main result (7.5) gives a range computable from the algebraic group structure of G, up to which j^* is an isomorphism. For a given group, it is rather small, about $\frac{1}{4}n$ for $\mathbf{SL}_n(\mathbf{Z})$ for instance; however, we may consider a sequence (G_n, Γ_n), where G_n and Γ_n are as G and Γ, for which our range tends to infinity, and of injective morphisms $f_n\colon (G_n, \Gamma_n) \to (G_{n+1}, \Gamma_{n+1})$. Since much is known about I_G, this allows us to determine the 'stable real cohomology' $H^q(\Gamma_n)$, n large, and $H^*(\underset{\rightarrow}{\rightarrow} \lim \Gamma_n)$ for sequences of classical groups (Section 11). For example, $H^*(\underset{\rightarrow}{\rightarrow} \lim \mathbf{SL}_n(\mathbf{Z}))\ldots$ is an exterior... algebra over infinitely many generators, one for each dimension of the form... $4i+1$ $(i = 1, 2, \cdots)$. The result for $\mathbf{SL}(\mathbf{Z})$ then implies that the dimension $K_i(\mathbf{Z}) \otimes \mathbf{Q}$ is one if $i \equiv 1 \bmod 4$ and zero otherwise $(i \geq 2)$."

More generally, the author computes $\dim(K_i\mathfrak{o} \otimes \mathbf{Q})$ $(i \geq 2)$ when $\mathfrak{o}$ is an order in a central division algebra D over an algebraic number field k. Indeed, if r_1 $[r_2]$ is the number of real [complex] places of k, then for $i \geq 2$, one has that $\dim(K_i\mathfrak{o} \otimes \mathbf{Q})$ is periodic of period four, and equals 0, $r_1 + r_2$, 0, r_2, depending on whether $i = 0, 1, 2, 3 \bmod 4$, respectively. (Also, applications to L-theory are given.) These results suggest a connection between the groups $K_i\mathfrak{o}$ and ζ_k, the Dedekind ζ-function of k (at least in the case when $\mathfrak{o}$ is the ring of integers in k). Thus the above result implies that if $i \geq 1$ is an integer then the order of the zero of ζ_k at $-i$ is equal to $\dim(K_{2i+1}\mathfrak{o} \otimes \mathbf{Q})$. This result is empirical, but S. Lichtenbaum [*Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic* (Proc. Conf., Seattle Res. Center, Battelle Memorial Inst., 1972), pp. 489–501, Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973] has formulated some striking conjectures concerning the possible relationship between K-groups and ζ-functions. Subsequent work by the present author, Lichtenbaum, and Coates has shed further light on this; in particular, see the author's subsequent article [Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. I, pp. 435–442, Canad. Math. Congr. Montreal, Que., 1975].

We quote further from the author's introduction: "The homomorphism j^* is defined for any discrete subgroup Γ of $G(\mathbf{R})$. If $G(\mathbf{R})/\Gamma$ is compact, then, as a simple consequence of Hodge theory, j^* is injective in all dimensions.... Results of Y. Matsushima [Osaka Math. J. **14** (1962), 1–20; MR0141138 (25 #4549)] show the existence of a constant $m(G(\mathbf{R}))$, determined by the Lie algebra of $G(\mathbf{R})$, up to which j^* is surjective. H. Garland had already shown [Ann. of Math. (2) **94** (1971), 534–548; MR0297733 (45 #6785)] that if $G(\mathbf{R})/\Gamma$ is only assumed to have finite invariant volume, and if every cohomology class in a given dimension $q \leq m(G(\mathbf{R}))$ is representable by a square integrable form, then Matsushima's arguments carry over without change and show that j^* is surjective in dimension q. Moreover, Garland and W. C. Hsiang [Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 354–360; MR0228504 (37 #4084)] gave, for Γ arithmetic, a range up to which this condition is fulfilled (a so-called 'square integrability criterion'). Our main concern is therefore the injectivity of j^* in the arithmetic case, but our discussion will also include the surjectivity and the square integrability criterion."

The author's proof rests on a geometric construction due to himself and J.-P. Serre [see #8337 above]. It suffices to assume Γ is torsion free. The construction of the author and Serre allows one to view X/Γ as the interior of a compact manifold with corners, $\overline{X}/\Gamma$. Also, assuming only that $G(\mathbf{R})/\Gamma$ has finite invariant volume, the author remarks that one can show that, "given an integer $m'(G)$, j^* is bijective at least up to $\min(m(G(\mathbf{R}), m'(G))$ if there exists a subcomplex C of ${\Omega_X}^\Gamma$ such that up to $m'(G)$: (i) $C \to {\Omega_X}^\Gamma$ induces an isomorphism in cohomology; (ii) $I_G \subset C$; (iii) C^q consists of square integrable forms".

To prove the existence of the subcomplex C, the author uses the compactification $\overline{X}/\Gamma$ mentioned earlier where, in fact, C is defined by means of certain growth conditions near the boundary of $\overline{X}/\Gamma$. These growth conditions will guarantee that (ii) is satisfied in all dimensions and that (iii) is satisfied up to a certain constant defined in terms of the $\mathbf{Q}$-roots of $\mathbf{G}$. As for (i), one introduces a presheaf $\mathcal{F}$ on $\overline{X}/\Gamma$. "For y on the boundary and U a neighborhood of y in $\overline{X}/\Gamma$, $\mathcal{F}(U)$ is the space of forms on $U \cap (X/\Gamma)$ satisfying the growth condition which is part of the definition of C. For $y \in X/\Gamma$, $\mathcal{F}(U)$ is simply the space of differential forms defined on U. This presheaf is a sheaf whose space of sections on $\overline{X}/\Gamma$ and X/Γ are C and ${\Omega_X}^\Gamma$, respectively. Since the inclusion $X/\Gamma \hookrightarrow \overline{X}/\Gamma$ is a homotopy equivalence, it suffices to prove that $\mathcal{F}$ is a fine resolution of $\mathbf{R}$ in $\overline{X}/\Gamma$. The 'fineness' is easy, so that the main point is to check a local Poincaré lemma near a boundary point, i.e., on a Siegel set. This is done by showing that the usual homotopy operator used to prove the Poincaré lemma on Euclidean space does not alter our growth condition."

From MathSciNet, January 2010

Howard Garland

MR0642850 (83h:22023) 22E41

Borel, Armand

Stable real cohomology of arithmetic groups. II.

Manifolds and Lie groups (Notre Dame, Ind., 1980), *pp.* 21–55, *Progr. Math.,* 14, *Birkhauser, Boston, Mass.*, 1981.

This paper is a sequel to one with the same title (but without the roman numeral) by the author [Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975); MR0387496 (52 #8338)]. We refer the reader to the review of that article (call it

SI) for background and here paraphrase part of the author's introduction to this sequel (call it SII).

Let G be a Zariski-connected isotropic reductive $\mathbf{Q}$-group, Γ an arithmetic subgroup of $G(\mathbf{R})$, and X the symmetric space for $G(\mathbf{R})$. Let I^q denote the space of Γ-invariant and G-invariant harmonic q-forms on X. In SI, a constant $d(G)$ was given such that the natural map $j^q\colon I^q \to H^q(\Gamma, \mathbf{C})$ is bijective for all $q \leq d(G)$. The three main steps of the proof were (i) the cohomology of Γ can be computed from a complex of differential forms satisfying a certain logarithmic growth condition at infinity; (ii) below a certain dimension $c(G)$, these forms are all square-integrable; (iii) below a certain dimension $m(G)$, j^q is injective and has as image the space of cohomology classes representable by square-integrable forms.

In SII, the case of nontrivial coefficients defined by a finite-dimensional complex representation E of G is also considered. Other growth conditions are also investigated, and it is proved that $H^*(\Gamma, E)$ can be computed using a complex of forms of moderate growth or a complex of weakly λ-bounded forms (where λ is a dominant linear form on the Lie algebra of a maximal $\mathbf{Q}$-split torus). The proof is similar to its analogue in SI. These forms are square-integrable up to a constant $C(G, \lambda, E)$, which constant is studied in great detail in SII. Vanishing theorems for relative Lie algebra cohomology with coefficients in infinite-dimensional unitary representations occurring in the spectrum of Γ, rather than the arguments originally due to Matsushima in SI, are used to improve the constant $m(G)$ and to provide an analogue $M(G, E)$ for nontrivial E. This leads to the bijectivity of j^q below a certain dimension and also to the vanishing of $H^q(\Gamma, E)$ below a certain dimension when E has no nontrivial subrepresentation.

Two related topics are treated in SII. (A) One can compute the cohomology with compact supports of $\Gamma\backslash X$ with a complex of fast decreasing forms or with a complex of weakly λ-bounded forms for $\lambda < 0$. It follows that a nonzero fast decreasing Γ-invariant harmonic form is not cohomologous to zero in $H^*(\Gamma, E)$ and is cohomologous to a closed form with compact support. This applies in particular to harmonic cusp forms. (B) The main theorem about bijectivity of j^q and vanishing of $H^q(\Gamma, E)$ remains valid for S-arithmetic groups and groups of rational points, with essentially the same bounds. The proof uses Bruhat-Tits buildings. As a result, the stability theorems of SI extend to S-arithmetic groups and groups of rational points, and their consequences for K-theory and L-theory are also valid for rings of S-integers and number fields.

From MathSciNet, January 2010

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