BOOK REVIEWS

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Yangians and classical Lie algebras, by Alexander Molev, Mathematical Surveys and Monographs, 143, American Mathematical Society, Providence, RI, 2007, xviii+400 pp., US$99 hardcover, ISBN 13: 978-0-8218-4374-1

In his fundamental 1985 paper [6], Drinfeld attached a certain Hopf algebra, which he called a *Yangian*, to each finite dimensional simple Lie algebra over the ground field $\mathbb{C}$. These Hopf algebras can be regarded as a tool for producing rational solutions of the quantum Yang-Baxter equation and are one of the main families of examples in Drinfeld's seminal ICM address [8] which marked the beginning of the era of quantum groups. For $\mathfrak{sl}_n(\mathbb{C})$, Drinfeld's Yangian embeds into a slightly larger Hopf algebra $Y(\mathfrak{gl}_n)$, the Yangian of $\mathfrak{gl}_n(\mathbb{C})$, which was discovered a few years earlier in the work on the quantum inverse scattering method by the St. Petersburg school. A few years later, Olshanski [14] introduced the *twisted Yangians* $Y(\mathfrak{so}_n)$ and $Y(\mathfrak{sp}_n)$ (assuming n is even in the latter case). These are certain subalgebras of $Y(\mathfrak{gl}_n)$ defined by "folding" the generators with respect to an approriate involution. The terminology here is confusing, as Olshanski's twisted Yangians $Y(\mathfrak{so}_n)$ and $Y(\mathfrak{sp}_n)$ are quite different from Drinfeld's Yangians associated to $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$; in particular, the former are not Hopf algebras. For the rest of this review, we are interested not with Drinfeld's Yangians in general, but just with the three families $Y(\mathfrak{gl}_n)$, $Y(\mathfrak{so}_n)$, and $Y(\mathfrak{sp}_n)$. The study of these algebras has revealed some hidden structure in the underlying classical Lie algebras, in the spirit of the sort of invariant theory to be found in Weyl's book [16].

Formally, the Yangian $Y(\mathfrak{gl}_n)$ can be defined as the associative algebra over $\mathbb{C}$ with generators $\{t_{ij}^{(r)} \mid 1 \le i, j \le n, r \ge 1\}$ subject to the relations

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} (t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}), \tag{1}$$

where $[x, y] = xy - yx$ is the commutator and $t_{ij}^{(0)}$ should be interpreted as the Kronecker δ_{ij}. The motivation behind these relations will be explained shortly, but first the reader should compare them to the familiar relations

$$[e_{ij}, e_{kl}] = \delta_{kj} e_{il} - e_{kj} \delta_{il}$$

satisfied by the matrix units $\{e_{ij} \mid 1 \le i, j \le n\}$ which generate the universal enveloping algebra $U(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. It follows that there are

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algebra homomorphisms

$$\mathrm{incl} : U(\mathfrak{gl}_n) \hookrightarrow Y(\mathfrak{gl}_n), \qquad e_{ij} \mapsto t_{ij}^{(1)} \qquad\qquad (1 \le i, j \le n),$$
$$\mathrm{eval} : Y(\mathfrak{gl}_n) \twoheadrightarrow U(\mathfrak{gl}_n), \qquad t_{ij}^{(1)} \mapsto e_{ij}, \quad t_{ij}^{(r)} \mapsto 0 \qquad (1 \le i, j \le n, r \ge 2),$$

such that the composition eval $\circ$ incl is the identity map. The second of these maps, the *evaluation homomorphism*, is of particular importance in the theory.

To explain the origin of the relations (1), introduce the generating function

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots ,$$

where u is a formal variable. Next gather all these generating functions together into a single *generating matrix* $T(u) = (t_{ij}(u))_{1 \le i,j \le n}$. Often it is convenient to regard the $n \times n$ matrix $T(u)$ instead as the tensor

$$T(u) = \sum_{i,j=1}^{n} e_{ij} \otimes t_{ij}(u) \in \mathrm{End}(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]],$$

where $\mathrm{End}(\mathbb{C}^n)$ is identified as usual with the algebra of $n \times n$ complex matrices. This gives us the flexibility to write down the more general expressions

$$T_1(u) = \sum_{i,j=1}^{n} e_{i,j} \otimes 1 \otimes t_{i,j}(u), \qquad T_2(u) = \sum_{i,j=1}^{n} 1 \otimes e_{i,j} \otimes t_{i,j}(u),$$

which are elements of the algebra $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]]$. Finally, letting $P \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the permutation $x \otimes y \mapsto y \otimes x$, we have *Yang's R-matrix*

$$R(u) = 1 - Pu^{-1} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)[[u^{-1}]],$$

which is the simplest nontrivial solution to the *quantum Yang-Baxter equation*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u),$$

where v is another indeterminate. Here, for $1 \le i < j \le 3$, $R_{ij}(u)$ denotes the operator in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)[[u^{-1}]]$ defined by applying $R(u)$ to the ith and jth tensors and the identity map in the remaining tensor position. With this notation, we can now write down the single matrix equation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v), \tag{2}$$

which by some formal manipulations is exactly equivalent to the defining relations of $Y(\mathfrak{gl}_n)$ recorded in (1). The equation (2) is an instance of the *R-matrix formalism* of Reshetikhin, Takhtajan and Faddeev [15], which is a basic tool in the quantization of various matrix-like structures.

When working with $Y(\mathfrak{gl}_n)$, it is usually essential to use the generating functions just introduced. To illustrate this philosophy, let us explain one basic result describing the center of the algebra $Y(\mathfrak{gl}_n)$. Consider the matrix

$$\begin{pmatrix} t_{11}(u) & t_{12}(u-1) & \cdots & t_{1n}(u-n+1) \\ t_{21}(u) & t_{22}(u-1) & \cdots & t_{2n}(u-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1}(u) & t_{n2}(u-1) & \cdots & t_{nn}(u-n+1) \end{pmatrix}. \tag{3}$$

Expanding each $(u-c)^{-r}$ as a formal power series in u^{-1}, the entries of this matrix belong to the algebra $Y(\mathfrak{gl}_n)[[u^{-1}]]$. Let

$$d_n(u) = 1 + d_n^{(1)} u^{-1} + d_n^{(2)} u^{-2} + \cdots \in Y(\mathfrak{gl}_n)[[u^{-1}]] \tag{4}$$

denote the column determinant of the matrix (3), by which we mean the usual expansion of its determinant as a sum of monomials, ordering the noncommuting terms in each monomial so that they appear in the same order as in the columns of the matrix. Often $d_n(u)$ is called the *quantum determinant*. For example:

$$\begin{aligned} d_2(u) &= t_{11}(u)t_{22}(u-1) - t_{21}(u)t_{12}(u-1) \\ &= 1 + \left(t_{11}^{(1)} + t_{22}^{(1)}\right)u^{-1} + \left(t_{11}^{(2)} + t_{22}^{(2)} + (t_{11}^{(1)}+1)t_{22}^{(1)} - t_{21}^{(1)}t_{12}^{(1)}\right)u^{-2} + \cdots . \end{aligned}$$

The infinite family of elements $d_n^{(1)}, d_n^{(2)}, \ldots$ are algebraically independent and generate the center of $Y(\mathfrak{gl}_n)$. The hardest part of this statement, the fact that each $d_n^{(r)}$ is actually central, has an elegant proof using the R-matrix formalism.

As an application of the result just formulated, we can give a nonclassical proof of a classical result giving explicit generators for the center of the algebra $U(\mathfrak{gl}_n)$. Multiply the cth column of the matrix (3) by $(u-c+1)$ for each $c = 1, \ldots, n$. Then apply the evaluation homomorphism to get the matrix

$$\begin{pmatrix} u+e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & u+e_{22}-1 & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & u+e_{nn}-n+1 \end{pmatrix} \tag{5}$$

with entries in $U(\mathfrak{gl}_n)[u]$. The classical *Capelli determinant* is the column determinant of this matrix. The result from the previous paragraph implies that the coefficients of $1, u, u^2, \ldots, u^{n-1}$ in this determinant all belong to the center of $U(\mathfrak{gl}_n)$. Finally, using the Harish-Chandra homomorphism, one checks routinely that these n elements are algebraically independent and generate the center of $U(\mathfrak{gl}_n)$. This analysis establishes in particular that the evaluation homomorphism $Y(\mathfrak{gl}_n) \twoheadrightarrow U(\mathfrak{gl}_n)$ maps the center of $Y(\mathfrak{gl}_n)$ surjectively onto the center of $U(\mathfrak{gl}_n)$.

Another classical construction which has found a new interpretation (and proof) in the context of $Y(\mathfrak{gl}_n)$ is the *Gelfand-Tsetlin basis* from [10]. This is a remarkable basis for any finite dimensional irreducible representation of $\mathfrak{gl}_n(\mathbb{C})$ consisting of simultaneous eigenvectors for the action of the *Gelfand-Tsetlin subalgebra* of $U(\mathfrak{gl}_n)$, that is, the commutative algebra generated by the centers of all the naturally embedded subalgebras $U(\mathfrak{gl}_m)$ for $m \leq n$. In order to define an analogous subalgebra in the Yangian, let us identify $Y(\mathfrak{gl}_m)$ with the subalgebra of $Y(\mathfrak{gl}_n)$ generated by the elements $\{t_{ij}^{(r)} \mid 1 \leq i, j \leq m, r \geq 1\}$, so that the evaluation homomorphism $Y(\mathfrak{gl}_n) \twoheadrightarrow U(\mathfrak{gl}_n)$ maps $Y(\mathfrak{gl}_m) \subseteq Y(\mathfrak{gl}_n)$ onto $U(\mathfrak{gl}_m) \subseteq U(\mathfrak{gl}_n)$. Then the Gelfand-Tsetlin subalgebra of $Y(\mathfrak{gl}_n)$ is the commutative subalgebra generated by the centers of all the subalgebras $Y(\mathfrak{gl}_m)$ for $m \leq n$. By the observation made at the end of the previous paragraph, the evaluation homomorphism maps the Gelfand-Tsetlin subalgebra of $Y(\mathfrak{gl}_n)$ surjectively onto the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_n)$. Unlike for $U(\mathfrak{gl}_n)$, the Gelfand-Tsetlin subalgebra of $Y(\mathfrak{gl}_n)$ is a free commutative algebra, with generators $\{d_m^{(r)} \mid 1 \leq m \leq n, r \geq 1\}$.

The Gelfand-Tsetlin subalgebra of $Y(\mathfrak{gl}_n)$ has another natural description. To explain this, we must first pass to a new set of generators arising from the *Gauss factorization* of the generating matrix $T(u)$. We can uniquely factor

$$T(u) = F(u)H(u)E(u)$$

for a lower unitriangular matrix $F(u)$, a diagonal matrix $H(u)$, and un upper unitriangular matrix $E(u)$. Let $h_i(u), e_j(u)$, and $f_j(u)$ be the ii-, $j(j+1)$-, and $(j+1)j$-entries of the matrices $H(u), E(u)$, and $F(u)$, respectively. The u^{-r}-coefficients $h_i^{(r)}, e_j^{(r)}$, and $f_j^{(r)}$ of these power series for $1 \leq i \leq n, 1 \leq j < n$, and $r \geq 1$ turn out to generate the algebra $Y(\mathfrak{gl}_n)$. It is even possible to write down a full set of relations for the new generators, which is closely related to Drinfeld's "new presentation" for Yangians discovered in [9]. The $h_i^{(r)}$'s, $e_j^{(r)}$'s, and $f_j^{(r)}$'s generate subalgebras $Y_0(\mathfrak{gl}_n)$, $Y_+(\mathfrak{gl}_n)$, and $Y_-(\mathfrak{gl}_n)$, respectively, such that multiplication

$$Y_-(\mathfrak{gl}_n) \otimes Y_0(\mathfrak{gl}_n) \otimes Y_+(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n) \tag{6}$$

is an isomorphism of vector spaces. This is the *triangular decomposition* of $Y(\mathfrak{gl}_n)$. Moreover the Gelfand-Tsetlin subalgebra of $Y(\mathfrak{gl}_n)$ is exactly the "diagonal" subalgebra $Y_0(\mathfrak{gl}_n)$. This coincidence is a consequence of a remarkable factorization

$$d_n(u) = h_1(u)h_2(u-1)\cdots h_n(u-n+1)$$

of the quantum determinant, which is another identity with a long history.

Since $Y(\mathfrak{gl}_n)$ has a triangular decomposition, it is not surprising that its finite dimensional irreducible representations can be classified in the spirit of highest weight theory. This was worked out by Drinfeld, building on results of Tarasov in the case of $Y(\mathfrak{gl}_2)$. The objects that parametrize the highest weights of the finite dimensional irreducible representations are known as *Drinfeld polynomials.* Rather than describe this classification in general, we want to round off this overview by focussing on one special family of finite dimensional irreducible representations, the so-called *skew representations* introduced by Cherednik [4]. These are the basic building blocks in the classification by Nazarov and Tarasov [13] of all the finite dimensional irreducible $Y(\mathfrak{gl}_n)$-modules on which the Gelfand-Tsetlin subalgebra acts semisimply.

To construct the skew representations of $Y(\mathfrak{gl}_n)$, we fix $m \geq 0$ and a pair of finite dimensional irreducible representations $V(\lambda)$ and $V(\mu)$ of $\mathfrak{gl}_{m+n}(\mathbb{C})$ and of $\mathfrak{gl}_m(\mathbb{C})$ parametrized by partitions λ and μ, respectively. By the Littlewood-Richardson rule, the space

$$V(\lambda \setminus \mu) = \mathrm{Hom}_{U(\mathfrak{gl}_m)}(V(\mu), V(\lambda))$$

is nonzero if and only if the Young diagram of μ is a subset of the Young diagram of λ, i.e., $\lambda \setminus \mu$ is a *skew Young diagram.* By standard theory [5, §9.1], $V(\lambda \setminus \mu)$ is irreducible as a module over the centralizer $U(\mathfrak{gl}_{m+n})^{U(\mathfrak{gl}_m)}$. Now, as well as the natural embedding considered so far, there is another embedding

$$\psi_m : Y(\mathfrak{gl}_n) \hookrightarrow Y(\mathfrak{gl}_{m+n}), \qquad h_i^{(r)} \mapsto h_{m+i}^{(r)}, \quad e_j^{(r)} \mapsto e_{m+j}^{(r)}, \quad f_j^{(r)} \mapsto f_{m+j}^{(r)},$$

whose image commutes with the naturally embedded subalgebra $Y(\mathfrak{gl}_m)$. This map ψ_m is quite nontrivial when written in terms of the usual $t_{ij}^{(r)}$ generators. Composing ψ_m with the evaluation homomorphism $Y(\mathfrak{gl}_{m+n}) \twoheadrightarrow U(\mathfrak{gl}_{m+n})$, we get a homomorphism $Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_{m+n})$ whose image is contained in the centralizer $U(\mathfrak{gl}_{m+n})^{U(\mathfrak{gl}_m)}$. The key point, which was the starting point for Olshanski's *centralizer construction* of the Yangian, is that $U(\mathfrak{gl}_{m+n})^{U(\mathfrak{gl}_m)}$ is actually generated by this image of $Y(\mathfrak{gl}_n)$ together with the center of $U(\mathfrak{gl}_m)$. This implies that $V(\lambda \setminus \mu)$ is also irreducible when viewed as a module over $Y(\mathfrak{gl}_n)$ via our homomorphism. This is the skew representation of $Y(\mathfrak{gl}_n)$. It has a Gelfand-Tsetlin basis indexed by skew Young tableaux, and the Gelfand-Tsetlin subalgebra acts semisimply on this basis. Thus the Yangian provides a natural setting that extends the classical Gelfand-Tsetlin construction from Young diagrams to skew Young diagrams.

We have not left enough space to discuss properly the twisted Yangians $Y(\mathfrak{so}_n)$ and $Y(\mathfrak{sp}_n)$. These were discovered by Olshanski by studying the centralizers $U(\mathfrak{so}_{m+n})^{U(\mathfrak{so}_m)}$ and $U(\mathfrak{sp}_{m+n})^{U(\mathfrak{sp}_m)}$, as in the previous paragraph. In fact, most of the topics touched upon above have analogues for the twisted Yangians. For instance, there is an explicit description of the centers of the twisted Yangians in terms of the *Sklyanin determinant* which, via the evaluation homomorphism, has led to the discovery of new formulae for generators of the centers of $U(\mathfrak{so}_n)$ and $U(\mathfrak{sp}_n)$. There is a classification of the finite dimensional irreducible representations by Drinfeld-like polynomials due to Molev [11], and there are analogues of the skew representations too. One of the most striking applications of the theory of twisted Yangians to date has lead to the construction of explicit weight bases for the finite dimensional irreducible representations of $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$, which are partial analogues of the Gelfand-Tsetlin bases for $\mathfrak{gl}_n(\mathbb{C})$.

Molev's book gives a detailed treatment of all of the topics discussed above, and many others, both for $Y(\mathfrak{gl}_n)$ and the twisted Yangians. It looks set to replace the foundational work [12] of Molev, Nazarov and Olshanski as the basic reference in this subject for the beginner. The new book goes much further into representation theory, which was not touched upon in [12], and incorporates many subsequent developments, several of which are due to Molev himself. It contains detailed and comprehensive proofs, though these tend to be rather technical algebraically in the twisted case. This subject has grown enormously in the twenty years since the introduction of Yangians, to the point that a mere 400 page text like this has to leave many interesting topics out. For instance, there is no discussion in the main body of the text of the Drinfeld functor relating the representation theory of $Y(\mathfrak{gl}_n)$ to that of the degenerate affine Hecke algebra (see [7, 1]) or of the connection between $Y(\mathfrak{gl}_n), Y(\mathfrak{so}_n)$, and $Y(\mathfrak{sp}_n)$ and finite W-algebras which has given further stimulus to this subject recently (see [3, 2]). However, at the end of every chapter, the author has included some valuable bibliographical notes in which these things are mentioned, as are many other applications of Yangians in both the mathematical and physical literature.

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