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Braid groups (With the graphical assistance of Olivier Dodane), by Christian Kassel and Vladimir Turaev, Graduate Texts in Mathematics, 247, Springer Science and Business Media, New York, 2008, xii+340 pp., ISBN 978-0-387-33841-5, hardcover, US$59.95

Ordering braids, by Patrick Dehornoy, with Ivan Dynnikov, Dale Rolfsen, and Bert Wiest, Mathematical Surveys and Monographs 148, American Mathematical Society, Providence, Rhode Island, 2008, x+323 pp., ISBN 978-0-8218-4431-1, hardcover, US$89.95

In the early 1970s, when this reviewer first began to study braids, they seemed to be outside the mainstream in geometric topology and were largely unrecognized in other areas of mathematics. However, as we write today, their importance, not just in topology but also in diverse other areas of mathematics, cannot be overestimated. Having been asked to review two new graduate-level books about braids, we would like to take advantage of the opportunity to speculate a little bit about some of the reasons for that change. On the way, and in our concluding remarks, we will describe how the textbook of Kassel and Turaev [27] and the research monograph of Dehornoy et al. [14] fit into the picture.

I. Historical background

We like to think of the mathematics of braids as having had its origins in the work of Gauss in the early 1800s. The recognition that the linking of two space curves had mathematical interest goes back to his work, and since Gauss was also thinking about braids, we start our story there.

The linking number $lk(K,K')$ of two oriented non-intersecting embedded loops $K,K' \subset \mathbb{R}^3$ is given by a very interesting double line integral that was discovered by Gauss (1777–1855):

$$\frac{1}{4\pi}\int_K\int_{K'}\left(\frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z'-z)(dxdy'-dydx')}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{3/2}}\right).$$

Gauss showed that the value of his integral does not change if $K\cup K'$ is deformed by an isotopy that is supported by an isotopy of $\mathbb{R}^3$, making it the first true *link type* invariant. We learned about this integral as an undergraduate from Exercise $14^\star$ on page 410 of [11], which asks the student to show that Gauss' integral, evaluated on the curve pairs in sketches (a) and (b) of Figure 1 is 0 and $\neq 0$. However, the vanishing of $lk(K,K')$ is not sufficient for two curves to be separated. At the end of Exercise $15^{\star\star}$ it is noted that $lk(K,K')$ vanishes on the curves in sketch (c), yet they are visibly inseparable. This very intuitive idea, that each component acts as an obstruction to the other, is the essence of linking. But sketch (d) shows a knot. The obstruction is still there, but Gauss' integral is not applicable because there is only one space curve. Sketch (e) in Figure 1 is also taken from Gauss' notebooks [37]. It illustrates a braid on four strands. Since he chose to draw a picture of a

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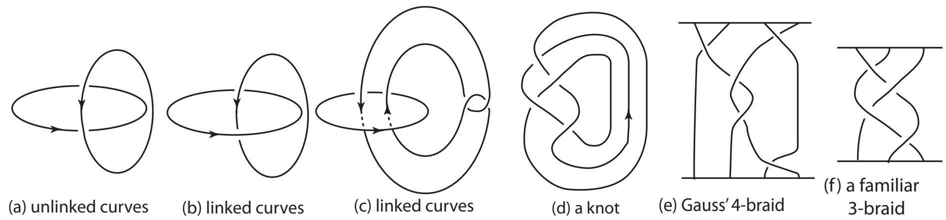

FIGURE 1. Links, knots and braids

4-braid rather than the traditional 3-braid that one finds in a person's hair (shown in sketch (f)), we speculate that he was thinking more generally about n-braids, and recognized that linking and braiding were related phenomena. Next to sketch (e), Gauss gave a method for describing his braid combinatorially. That is one reason for passing from knots and links to braids, although there are also other ways to describe knots and links combinatorially, for example by their *Gauss diagrams* (a topic for another day).

In 1891, braids appeared in a different guise, in the work of Hurwitz [24]. Hurwitz's braids were motions of n distinct points $p_1(t), \dots, p_n(t)$ on the 2-sphere, where the motion is required to return the set of points to its initial position, possibly permuting them; also, at any fixed $t \in [0,1]$ the points are distinct. Follow the motion of the n points, and a braid appears in $\mathbb{S}^2 \times [0,1]$. This interpretation is the reason some say that the study of braids was initiated in 1891 by Hurwitz.

The first attempt at a careful definition of a braid on n strands and the discovery that braids form a *group* came in 1925, with the publication of the foundational paper of E. Artin [3]. The group operation is realized by the concatenation of two braid patterns followed by rescaling. Inverses are realized by reflection in the plane that contains the endpoint. The identity is represented by n straight lines. The equivalence relation that makes all this work is level-preserving isotopy of one braid to another through braids, keeping the initial and final points of every strand fixed. (We will define it more precisely in a moment.) Artin proved that the group B_n of braids on n strands is finitely presented, with generators that are depicted geometrically in Figure 2.

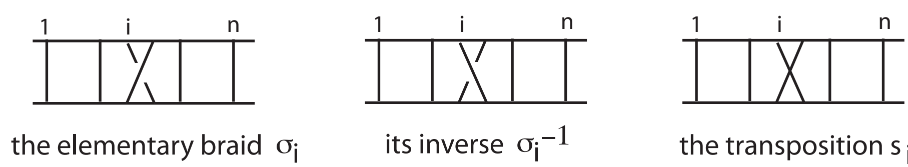

FIGURE 2. Elementary braids and permutations

He discovered the presentation,

$$\begin{aligned} \langle\, \sigma_1, \dots, \sigma_{n-1}\ ;\ \ \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1,\ 1 \leq i,j \leq n-1 \rangle, \end{aligned} \tag{1}$$

and noticed that the symmetric group admits a closely related presentation, with generators $s_1, \dots, s_{n-1}$ and relations $s_is_j = s_js_i$ if $|i-j| \geq 2$; also $s_is_js_i = s_js_is_j$

if $|i-j| = 1$, and in addition $s_i^2 = 1$ for every $i = 1, \ldots, n-1$, establishing a homomorphism from B_n to the symmetric group Σ_n sending $\sigma_i \to s_i$. What he did not appreciate is that a great deal of the theory of finite-dimensional representations of Σ_n, using the language of partitions and Young diagrams, generalizes to representations of B_n. We give one almost naive example [9]: the reducible representation of Σ_n over the ring $\mathbb{Z}$ of integers that sends s_i to the $n \times n$ matrix $\mathrm{I}_{i-1} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \mathrm{I}_{n-i-1}$ lifts to a reducible representation of B_n over the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ sending $\sigma_i \to \mathrm{I}_{i-1} \oplus \left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right) \oplus \mathrm{I}_{n-i-1}$. Thinking of t as a complex parameter, this representation of B_n collapses to a faithful representation of Σ_n when $t = 1$.

Thus braids appeared, from the start, in several guises:

Example i. As a *group of motions* of n distinct points on a manifold;

Example ii. As *geometric braids*, embedded in the space between two parallel planes, with multiplication defined by juxtaposition and rescaling;

Example iii. As the preimage of a homomorphism onto the well-known *group of permutations*, a group whose structure contains (as every undergraduate knows) the keys to finite group theory.

In fact, there are many other interpretations of B_n, some quite subtle, which taken together suggest that braiding is a very fundamental mathematical and physical phenomenon. Thus a new and accessible graduate level textbook [27] on the subject is very welcome.

II. Braiding is fundamental

There is a succinct way to define Artin's braid group and its close relative, Hurwitz' group, and at the same time to generalize both. Let $M = M^k$ denote a k-dimensional manifold. The *configuration space* $\mathcal{C}_n(M)$ is $\{(p_1, \ldots, p_n), \quad p_i \in M, \quad p_i \neq p_j \text{ if } i \neq j\}$. (More generally, the points p_i can be replaced by appropriate codimension-2 submanifolds of M, leading to still more generalized braiding.) It is a subset of $\prod_n M$. The *pure braid group* $P_n(M)$ is the fundamental group $\pi_1(\mathcal{C}_n(\mathbb{M}, \star))$, where $\star = (p_{1,\star}, \ldots, p_{n,\star})$. The symmetric group Σ_n acts on $(\mathcal{C}_n(M), \star)$ on the left, sending $(p_1, \ldots, p_n) \to (p_{\mu_1}, \ldots, p_{\mu_n})$. We may think of $(\mathcal{C}_n(M))/\Sigma_n$ as the space of unordered n-tuples of points on M. The n-strand *braid group* $B_n(M)$ of the manifold M is defined to be the fundamental group $\pi_1((\mathcal{C}_n(M), \star)/\Sigma_n)$. By definition, the two groups $P_n(M)$ and $B_n(M)$ are related by the short exact sequence $\{1\} \to P_n(M) \to B_n(M) \to \Sigma_n \to \{1\}$, so for every manifold M there is a natural homomorphism $\pi : B_n(M) \to \Sigma_n$, and $P_n(M) = \ker(\pi)$.

It is immediate that $B_n(\mathbb{S}^2)$ is Hurwitz's group of motions of n points on $\mathbb{S}^2$, that is, Example i above. Each element of $B_n(\mathbb{S}^2)$ is defined by n continuous functions $[0,1] \to \mathbb{S}^2$, where the trace of the ith function $p_i(t)$ starts at the ith base point, say z_i and ends at z_{μ_i}. The fact that $p_i(t) \neq p_j(t)$ if $i \neq j$ makes it possible for us to visualize the motion as a simultaneous motion of n points on $\mathbb{S}^2$, and even more as an element of $\mathrm{Diff}(\mathbb{S}^2, \star)$, defined up to base-point-preserving isotopy. There is a vast literature about braid groups of 2-manifolds, and there are also higher-dimensional analogues of motion groups. That is, instead of looking at n-tuples of points in M, think of special subsets of $\mathcal{C}(M)$ and their motions in M.

Now let M be the complex plane $\mathbb{C}$. We claim that the group $B_n = B_n(\mathbb{C})$ is Artin's braid group, discussed as Example ii above. We visualize braids as lying in the slice of 3-space $\mathbb{C} \times [0,1]$; see Figure 1(e) and 1(f) and Figure 2. The ith braid strand is the image of $[0,1]$ under the function $p_i : [0,1] \to \mathbb{C} \times [0,1]$. It begins at $z_i \in \mathbb{C} \times \{0\}$ and ends at $z_{\mu(i)} \in \mathbb{C} \times \{1\}$. The fact that $p_i(t) \neq p_j(t)$ if $i \neq j$ expresses the facts that braid strands cross every intermediate plane $\mathbb{C} \times \{t\}$ in n distinct points. Configuration spaces also give us the homomorphism $B_n \to \Sigma_n$, discussed in Example iii above. One is of course led to study the kernel, i.e., the pure braid group $P_n(\mathbb{C})$. Its structure, as a sequence of semi-direct products of free groups of decreasing rank, was studied and uncovered in [3].

Example iv. We next describe several ways that braids arise naturally in the study of algebraic functions. Let $f(z) = \sum_{j=1}^{n} w_{n-j} z^j$ be a monic polynomial of degree n with complex coefficients in one complex variable z. Assume that its roots $(z_1, \ldots, z_n)$ are all distinct, so that the array of roots defines a point in the configuration space $(\mathcal{C}_n(\mathbb{C}))/\Sigma_n$. Let $\pi_1, \ldots, \pi_n$ be the elementary symmetric functions. Regarding $\pi_1, \ldots, \pi_n$ as functions of the roots and knowing that each π_i is invariant under the action of Σ_n, it is seen that they induce a root-to-coefficient map $\Phi : \mathcal{C}_n(\mathbb{C})/\Sigma_n \to \mathbb{C}^n$, with $\Phi(z_1, \ldots, z_n) = (w_1, \ldots, w_n)$. The map Φ is invertible in the complement of the set of points in the coefficient space where two or more roots coalesce. Allowing the roots to move along a path $(z_1(t), \ldots, z_n(t))$, where $t \in [0,1]$ and $z_i(t) \neq z_j(t)$ if $i \neq j$, also $(z_1(1), \ldots, z_n(i)) = (z_{\mu_1(0)}, \ldots, z_{\mu_n(0)})$, the path of roots determines a braid. Out of the enormous literature in the area, we single out a small number of investigations into this beautiful geometry:

- Following [32], let $f(w,z)$ be a polynomial in two complex variables, and consider the surface $S = \{(w,z) \in \mathbb{C}^2 \mid f(w,z) = 0\}$. Assume that (w_0, z_0) is an isolated singular point, and let $\mathbb{B}_\epsilon$ be a 4-ball neighborhood of (w_0, z_0). Then the intersection of S with $\partial \mathbb{B}_\epsilon = \mathbb{S}^3$ is a 1-manifold, i.e., a link in $\mathbb{S}^3$. The links that arise in this way are called *algebraic links*. They are a special set of iterated torus links, and, by our earlier discussion, they have natural presentations as closed braids. These links have received much attention by knot theorists. For example, they are fibered, and the monodromy of the fibration is known, as is the Seifert form. See the monograph [16] for a wealth of information.
- Rudolph [38] singles out a class of links that are the closure of braids which are products of conjugates of $\sigma_1, \ldots, \sigma_{n-1}$. He calls them *quasi-positive braids*. He then allows the coefficients of $f(z)$ to be functions of a variable w and studies *n-valued algebraic functions* $f(w,z) = \sum_{j=0}^{j=n} f_j(w) z^{n-j}$, where the equation $f(w,z) = 0$ is required to have n distinct solutions $z_1, \ldots, z_n$, each being an analytic function of w. He gives a construction that goes from algebraic functions without poles to braids. One might wonder whether every link arises in this way as a closed braid. However, Rudolph proves that the braids are all quasi-positive and shows that there are links which have no representatives that are closed quasi-positive braids, making this a new and interesting construction.
- S. Orevkov [34, 35] initiated investigations into the relationship between Hilbert's 16th problem and the theory of braids. He made use of Rudolph's work.
- Gorin and Lin [23] regarded braids as the start-point for a way to generalize Galois theory. We single out, from their very many contributions, an open problem about braids which they solved in the context of complex function theory: the structure of the commutator subgroup of B_n was first uncovered in [23].

Example v. In 1968 a very different connection between braids and permutations was discovered by F. Garside [20]. Observe that Artin's presentation (1) for B_n also may be thought of as a presentation for the *positive braid monoid* B_n^+. Garside's new idea was to focus on the monoid B_n^+, examining the ways in which it differs from the group Σ_n and its relationship to B_n. Let Δ_n be a half-twist of all of the braid strands. Garside proved that every element $x \in B_n$ could be represented in a unique way as $\Delta_n^q X$, where $q \in \mathbb{N}$ and $X \in B_n^+$, with $X \neq \Delta_n Y$ for any $Y \in B_n^+$. Also, that Δ_n is *almost* in the center of B_n, so that it can be moved about freely in a braid word. He proved that the natural map $B_n^+ \to B_n$ is an embedding. That is, two positive braid words define the same element of B_n if and only if they define the same element of B_n^+. Later, Thurston [17] and also Adyan [1] showed that every element $X \in B_n^+$ admits a unique normal form $X = X_1 X_2 \cdots X_s$, where each X_i is a special type of braid in B_n^+ that is determined uniquely by the permutation of its strands. Moreover, each X_i is a maximal *permutation braid* in the product $X_i X_{i+1} \cdots X_s$. The longest *permutation braid* is Δ_n, the start of a new story which we have no room to discuss, save to note that the longest reduced word in the generators $s_1, \ldots, s_n$ of Σ_n generalizes to the *Coxeter element* in certain Coxeter groups.

Garside's discovery was the start of a new area of mathematical research. There is a connection between Coxeter groups and a class of groups that has become known as *Garside groups* [13]. Just as the symmetric group Σ_n is the canonical example of a Coxeter group, so the braid group B_n, with its associated monoid B_n^+, is the canonical example of a Garside group. The underlying mathematics in this work is of course related to the study of the kernel of the natural homomorphism from a Garside group to its associated Coxeter group, but the emphasis is on the passage from a Garside monoid to its associated Coxeter group and not on the homomorphism or its kernel.

Example vi. The mathematics of *configuration spaces* may be regarded as a subspecialty in its own right. See, for example, (a) the third and (b) the fourth chapter of [5] for tutorial-type reviews of the ways that configuration spaces appear in (a) homotopy theory and (b) robotics.

Example vii. One of the great problems in mathematics is the determination of the homotopy groups $\pi_n(S^k)$, and as it turns out, their determination is related in multiple ways to problems about braids [6]. In the case $k = 2$, a great deal is known about the groups $\{\pi_{n-1}(\mathbb{S}^2), n \in \mathbb{N}\}$, but what is missing is a uniform approach to understanding them, and the study of braids offers exactly that. A *Brunnian braid* $Br_n(M)$ on a manifold M is a braid in $B_n(M)$ that becomes trivial after removing any one of its strands. The groups $\mathrm{Br}_n(\mathbb{D}^2)$ and $\mathrm{Br}_n(\mathbb{S}^2)$ are subgroups of the pure braid groups $P_n(\mathbb{D}^2)$ and $P_n(\mathbb{S}^2)$. Let $\iota : \mathbb{D}^2 \to \mathbb{S}^2$ be the canonical inclusion map. The following exact sequence is established in Theorem 1.2 of [6] when $n \geq 5$:

$$\{1\} \longrightarrow \mathrm{Br}_{n+1}(\mathbb{S}^2) \longrightarrow \mathrm{Br}_n(\mathbb{D}^2) \xrightarrow{\iota_*} \mathrm{Br}_n(\mathbb{S}^2) \longrightarrow \pi_{n-1}(\mathbb{S}^2) \to \{1\}. \tag{2}$$

The sequence (2) says, essentially, that any non-trivial element in $\pi_{n-1}(\mathbb{S}^2)$ can be represented by a Brunnian braid over $\mathbb{S}^2$ that is not Brunnian over $\mathbb{D}^2 \subset \mathbb{S}^2$. Since generators for image(ι_*) are now known [4], the remaining problem to the sought-for uniform approach to understanding $\pi_{n-1}(\mathbb{S}^2)$, $n \geq 5$, is to obtain generators for $\mathrm{Br}_n(\mathbb{S}^2)$.

Example viii. The fact that braids play a central role in our understanding of knots and links in $\mathbb{S}^3$ is so well known, with many textbooks and monographs giving accounts of various aspects (for example, see [27], [10], [31]) that we were inclined to omit any discussion of knotting and linking as an example of an application of braids; however, a few ways in which our knowledge of knots and braids has really changed over the past 30 years should be mentioned.

A *diagram* for a knot or link is a planar projection of the link onto a plane in which the singularities are at most finitely many transverse double points, with a marking to indicate the overcrossing branch. In 1923, J.W. Alexander [2] proved that every link $\mathcal{L}$ could be represented non-uniquely as a closed braid. This meant, to Alexander, that if $\mathbb{R}^3$ is given cylindrical coordinates (r, θ, z), then there is a parametrization of each component of $\mathcal{L}$ such that as one traverses the component, $d\theta/dt \neq 0$. Call the associated diagram on the plane $z = 0$ an *Alexander diagram*. These diagrams seem quite special, but in 1987, S. Yamada [41] saw how to generalize them: Consider an arbitrary diagram and eliminate the double points by smoothing them, preserving orientation. The diagram then goes over to a family $\mathcal{S}$ of pairwise disjoint oriented simple loops called *Seifert circles*. Call two Seifert circles $C, C' \in \mathcal{S}$ *coherent* if they represent the same element of $H_1(A)$, where A is the annulus they cobound. A link diagram is a *Yamada diagram* if all of its Seifert circles are coherently oriented. It is easy to see that {Alexander diagrams} $\subset$ {Yamada diagrams}. Yamada proved that his diagrams are essentially closed braid diagrams, even though they do not satisfy the Alexander criterion. P. Vogel [40] went a step further, when he showed algorithmically that many diagrams which fail the Yamada criterion can be modified in a very simple way to Yamada diagrams.

The versatile *rectangular diagrams* that I. Dynnikov introduced in [15] give yet another way to find braids in disguise. Thus, while closed braids had once been thought to be a very restricting way of studying knots and links, that is no longer the case.

Example ix. Surely automorphic forms are, in principle, about as far as one can get in mathematics from braids and knots. A blockbuster example, which arose out of the beautiful insights of Etienne Ghys in [21], demonstrates the underlying unity in mathematics, as well as the ubiquity of braids:

(a) In [21], Ghys defined a flow Φ_t (he called it the *modular flow*) on $\mathbb{S}^3 \setminus \mathcal{T}$, where T is the trefoil knot. Its closed orbits lie in $\mathbb{S}^3 \setminus \mathcal{T}$, and so can be regarded as knots (he called them *modular knots*) in $\mathbb{S}^3$. This was a very new idea! Regarding them as knots in $\mathcal{S}^3$ allowed him to interpret the *Rademacher function*, a mapping from conjugacy classes in PSL(2,$\mathbb{Z}$) to the integers $\mathbb{Z}$, as the Gauss linking number between modular knots and the *missing trefoil* $\mathcal{T}$. See [22] for an article on this mathematics that is directed at a broad audience, using elegant computer graphics to bring the topic to life.

(b) Switching momentarily to a new topic, the *Lorenz flow* Ω_t on $\mathbb{R}^3$ had been studied intensively in dynamical systems as a prototypical example of a chaotic dynamical system. Its orbits are obtained by integrating a system of three nonlinear ordinary differential equations in three space variables x, y, z and time t. See [8], where the Lorenz equations are given and their closed orbits are studied, as knots and links.

(c) Returning to [21], Ghys proved that the closed orbits in the flow $\Phi_t : \mathbb{S}^3 \setminus \mathcal{T} \to \mathbb{S}^3 \setminus \mathcal{T}$ are in one-to-one correspondence with the closed orbits in $\Omega_t : \mathbb{R}^3 \to \mathbb{R}^3$. Lorenz knots in $\mathbb{R}^3$ have a natural representation as closed braids, and Ghys' correspondence respected this closed braid representation. He also established the position of the missing trefoil $\mathcal{T}$ in the modular flow, and followed it during his passage from modular knots to Lorenz knots. Using the work in [8], the linking numbers in (b) above suddenly became easy to compute. It then follows from Ghys' work that the Rademacher function in (a) above is also easy to compute.

(d) The solutions to the Lorenz equations have a very sensitive dependence on initial conditions. Therefore two points that are arbitrarily close in $\mathbb{R}^3$ but belong to distinct closed orbits in the Lorenz flow can have very different long-term behavior. This suggests that there might a simple explanation why the Rademacher function is so difficult to understand, namely that the modular flow has a similar sensitive dependence on initial conditions, so that points in the flow which are close in $S^3 \setminus \mathcal{T}$ might have very different linking numbers with $\mathcal{T}$. Here we bipass a difficulty, namely that Ghys has proved that the orbits in the two flows coincide. But he has not proved that the flows coincide. We conjecture that they do.

(e) If, in fact, the two flows coincide, one wonders what happened to the missing trefoil in the Lorenz flow? Recently we learned that T. Pinsky [36] showed experimentally that the missing trefoil is indeed present in the Lorenz flow. Her insight in this regard makes it immediately clear how and why this curious fact had been missed for 50 years, but we leave this to her to explain, when she writes her paper.

Example x. We have ranged far and wide in our discussion of braids, without even mentioning the main topic in the second book under review [14], and it is long past time to do so. A group G is *left orderable* if there is a total order $<$ on the elements of G which is invariant under left multiplication. That is, for all $h, k \in G$ one and only one of the following holds: $h < k, h = k$, or $k < h$, and also for every $g \in G,\ h < k \Longrightarrow gh < gk$. Left orderability implies right orderability (even though the two orders are in general different), so we simplify the notation and say that a group is *orderable* if it is left or right orderable, and *biorderable* if an ordering exists which is both a left and a right ordering.

The discovery in 1994 [12] that B_n is orderable (but not biorderable) and the flood of work that followed it was certainly one of the major developments in braid theory in recent years. The *Dehornoy* or *σ-ordering* is based upon the following criterion: Using the presentation (1), a braid is greater than the identity braid if and only if, for some i, it can be written in the form $\beta_0\sigma_i\beta_1\sigma_i\cdots\sigma_i\beta_k$, where $\beta_0, \dots, \beta_k$ use only the letters $\sigma_{i+1}^{\pm1}, \dots, \sigma_{n-1}^{\pm1}$. Alas, the proof in [12] that B_n is orderable, based upon properties of left-distributive systems, seems to have been unnoticed outside the community of logicians until the five authors of paper [19] gave their own proof that it is an ordering: Think of B_n as the mapping class group of the n-times punctured disc with admissible maps fixing $\partial\mathbb{D}^2$. Assume that $\mathbb{D}^2 = \mathbb{D}^2_+ \cup \mathbb{D}^2_-$ is the unit disc, with the punctures on $\mathbb{D}^2_+ \cap \mathbb{D}^2_- \subset x$-axis. Choose $\beta \in B_n$. The ordering arises from a careful study of precisely how the half-disc $\beta(\mathbb{D}^2_-)$ intersects $\mathbb{D}^2_-$ and in particular how $\beta(\mathbb{D}^2_+ \cap \mathbb{D}^2_-)$ crosses the x-axis. After that came the explosion of ideas that are discussed in [14].

A curious part of this history is that a few years after [19] appeared, it was shown in the article [39] that there is a very simple proof that B_n is orderable, different from the argument in [19], based on the hyperbolic structure on $\mathbb{D}^2$. It could have been (but was not) discovered by Jakob Nielsen. It is attributed to Thurston, and like many other proofs at that time it seems to have not been written down anywhere. Thus it could be said that after Thurston's work on surfaces in the 1980s put Nielsen's massive work [33] into a unified context, it became well known (to those who talked to Thurston) that B_n is orderable.

In this regard we note that a two-sentence proof that B_n is torsion free is possible after one knows that B_n is orderable, vis: Assume there exists a braid $\beta \in B_n$ with $1 < \beta$ and $\beta^p = 1$. It would follow that $\beta^k < \beta^{k+1}$ for every positive k. However, $1 < \beta < \beta^2 < \cdots < \beta^{n-1} < \beta^n = 1$, a contradiction. We mention this because the first proof that B_n is torsion free was given in [18], and it uses a fair amount of sophisticated mathematics. Since this writer is familiar with several major unsuccessful attempts to simplify it, the elementary proof we just gave, and indeed the fact that B_n is orderable, was missed entirely. More seriously, the very beautiful topology that was shown in [19] to underly the Dehornoy ordering had also been missed.

III. Concluding remarks

The book of Kassel and Turaev is a textbook [27] for graduate students and researchers. As such, it covers the basic material on braids, knots, and links in Chapters 1 and 2 at a level which requires minimal background, yet moves rapidly to non-trivial topics. After that, representations of the braid group are handled with a thoroughness that is appropriate, given the authors' own research: They are introduced in Chapter 3 with the classical Burau representation and the more recent representations of Lawrence, Krammer, and Bigelow, including proofs of non-faithfulness of the former and faithfulness of the latter. Chapters 4 and 5 are devoted to the Iwahori–Hecke algebra representations and knot polynomials. These topics are especially appropriate because both authors have made significant contributions to recent research on quantum representations of B_n and its applications to the study of knots and 3-manifolds. The book concludes with two somewhat newer topics, treated in a very nice introductory fashion, namely Garside Monoids (Chapter 6) and Braid Ordering (Chapter 7). It is a carefully planned and well-written book; the authors are true experts, and it fills a gap. We expect it will have many readers.

The monograph [14] is directed at researchers. The σ-ordering is explored with a thoroughness that reveals many new aspects of the geometry of braids. Its style is very appealing to this reviewer. Every new idea is introduced pleasantly, well-motivated intuitively, and proved with care. The fact that braid ordering is a limited topic means that the covering could be (and is) exhaustive, but of course only up to the date of publication. Therefore, it stopped short of [25], where it is shown that the Dehornoy ordering is related to the *braid foliations* that were introduced by this reviewer and W. Menasco in [7]. Since we are particularly fond of that mathematical structure, we were pleased when it made a most unexpected appearance in new discoveries that relate to the Dehornoy ordering of B_n.

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