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MODELS OF DISCRIMINANT SURFACES

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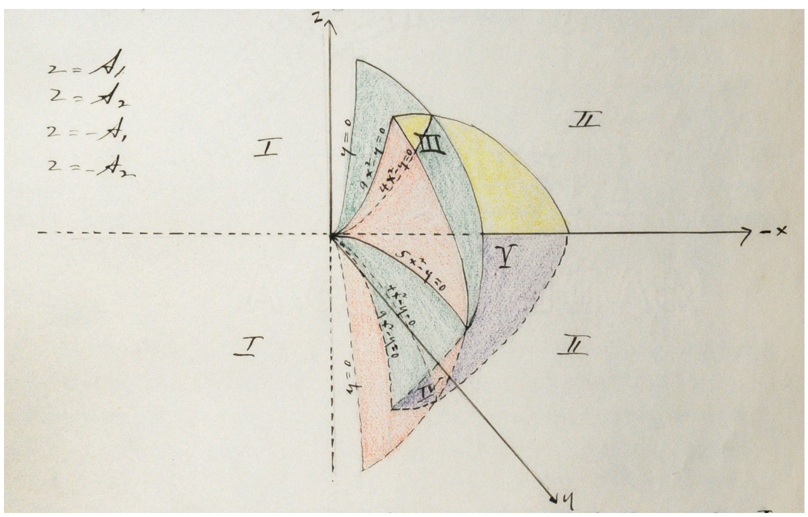
1. Introduction



The illustration[1] shown here appears in the master's thesis of Mary Emily Sinclair (1878–1955). Her thesis, supervised by Oscar Bolza at the University of Chicago in 1903, deals with quintic polynomials $p(t) = t^5 + xt^3 + yt + z$. Every (real) triple (x, y, z) yields such a polynomial, and Sinclair describes the sets of all (x, y, z) such that the corresponding polynomial has a given number of real zeros. For this, one needs the set of all (x, y, z) such that $p(t)$ has a multiple zero.

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[1]The illustration used above and on the cover was provided courtesy of Oberlin College Library Special Collections.

This set is called the *discriminant surface* of the family of polynomials and is shown in Sinclair's sketch.

Discriminant varieties are still studied in algebraic geometry, differential geometry, dynamical systems (catastrophy theory, bifurcation theory, singularity theory, ...). Although for more than a century the geometry of discriminants has been considered a topic in pure mathematics, it also has a relation with engineering mathematics. Namely, a discriminant surface consists of all (x, y, z) for which a t exists with $p(t) = p'(t) = 0$. For given t this system of two linear equations in x, y, z defines a line. The union of these lines is the discriminant surface. This observation is used to show that the points of the discriminant surface contained in any such line all have a common tangent plane. This implies that at least infinitesimally, one can start with a flat surface and roll it into the shape of a discriminant surface (it is a so-called developable surface). Applications of this appear in ship design, architecture, etc. One of the earlier texts on the subject [MBCZ], published in 1877, was written by two civil engineers and supplemented with an appendix by the geometer H. G. Zeuthen.

2. Kerschensteiner and Schoute

In 1864, J. J. Sylvester published a paper [Sy64] in which he treats the question of how many real zeros a given polynomial has. The novelty of his approach is geometry: a polynomial is a point, the polynomials with a given zero form a hyperplane, and the ones with a multiple zero are a determinant hypersurface. Sylvester's idea is picked up by various others including Leopold Kronecker and Felix Klein. In 1892, upon the request of the DFG (German Mathematical Society), W. Dyck published a catalog [Dy93] of models and instruments used in mathematics and physics. Klein's preface [Kl92] to this discusses the geometry of the discriminant.

He explains Sylvester's ideas, which he visualizes for families of polynomials depending on two variables x, y. The high school case t^2+xt+y is in fact his leading example. The shape of the discriminant curve suffices to determine which (x, y) lead to polynomials with a given number of real zeros. He ends his exposition with the remark that the same should be possible for families of polynomials depending on three variables x, y, z. Here one needs the discriminant surface, and concerning the solution of his problem, Klein states:

> *Hier ist offenbar ohne geeignete Modelle nicht durchzukommen. Es wird sehr dankenswert sein, wenn jemand die Herstellung solcher Modelle in die Hand nehmen wollte.*[2]

The pages 168–173 of the same catalog, written by the German *Gymnasiallehrer* (teacher at a grammar school) Georg Kerschensteiner (1854–1932), discuss the discriminant surfaces corresponding to $t^3 + xt^2 + yt + z$ and to $t^4 + xt^2 + yt + z$. Kerschensteiner does not mention actual models, but he includes two detailed illustrations; see Figures 1 and 2. Sketches resembling these figures were reproduced in, for example, H. Weber's *Lehrbuch der Algebra* [We95, §78] and in F. Klein's celebrated *Elementary Mathematics from an Advanced Standpoint* [Kl32].

[2] Translation: Evidently this cannot be achieved without suitable models. We would be very much indebted to anyone willing to take up the construction of such models.

Ausdrack der Discriminantenfläche:
$\Delta_3^*) \equiv [c + a(2a^2 - 3b)]^2 + 4(b - a^2)^3 = 0$

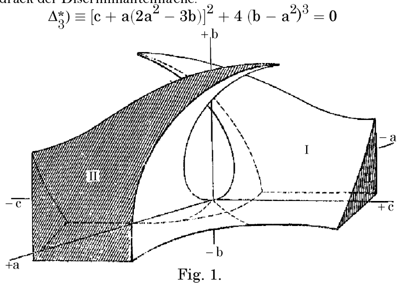

Fig. 1.

Die Fläche teilt den Raum in 3 Räume R_I R_{II} R_{III}, für deren Punkte

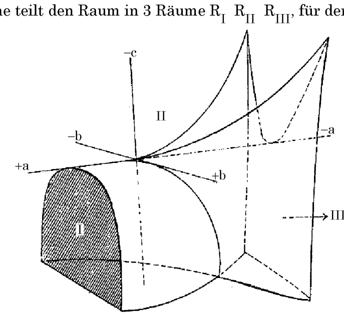

Fig. 2.
die Gleichung G4 = o der Reihe nach: 0, 2, 4 reele Wurzeln besitzt.

Kerschensteiner later became a school reformer and pedagogist; his ideas concerning vocational schools have had a large influence on the German school system and brought him worldwide recognition.

Klein's call in Dyck's catalog inspired the Dutch geometer Pieter Hendrik Schoute (1846–1913) from Groningen to construct string models of discriminant surfaces. At the monthly science meeting of the Royal Dutch Academy of Sciences (KNAW) in Amsterdam on Saturday, May 27, 1893, he presented and described three models. Biographical notes concerning Schoute are presented in [Sch]. Schoute's explanation of his models appeared in the minutes (in Dutch) of the KNAW meetings [KNAW, pp. 8–12] in 1894 and (in German) in 1893 in a supplement (*Nachtrag*) to Dyck's catalog [Dy93].

The three families of polynomials that Schoute considers are

$$t^3 + xt^2 + yt + z \quad \text{and} \quad t^4 + xt^2 + yt + z \quad \text{and} \quad t^6 - 15t^4 + xt^2 + yt + z.$$

For these cases he exhibits the singular points of the discriminant surface, and he explains in detail how he designed the actual models. As of this writing, they are still on display in the mathematics institute of the University of Groningen.

Schoute explains why he considers polynomials of degrees $3, 4$, and 6: the points in space not on the surface correspond to real polynomials with only simple zeros. For degree 3, there are two possibilities for the number of real zeros, so the surface partitions space into two parts depending on this number of zeros. In the case of degree 4, there are three possibilities for the number of real zeros (which obviously all occur in the given family of polynomials). Degree 6 is the smallest case with four possibilities for the number of real zeros, and Schoute chooses a particular family of sextics in which all four possibilities occur.

During the next KNAW meeting (Saturday, June 24th, 1893), Schoute informed his colleagues that

> *Dr. Kerschensteiner also constructed models of discriminant surfaces, namely by moulding together tinplate cross sections.*

(Compare [KNAW, p. 44] for the original text in Dutch.)

The correspondence between Schoute and Kerschensteiner probably inspired the latter to write a text [Ke93] published in the supplement of Dyck's catalog [Dy93]. Here he treats the family

$$t^5 + xt^2 + yt + z.$$

The discriminant surface in this case partitions real 3-space into only two parts, much like what also happens for polynomials of degree 3.

3. Sinclair and Hartenstein

Probably after recommendations by Klein, the German firm Martin Schilling in 1908 decided to produce some discriminant surfaces, as Series XXXIII of their collection of mathematical models for higher education. According to Schilling's catalog [S11], the three models in Series XXXIII were designed by Mary Emily Sinclair and Roderich Hartenstein.

Biographical notes concerning Mary Emily Sinclair (1878–1955) can be found in [GL08]. Her master's thesis [Si03], completed at the University of Chicago in 1903 and supervised by Klein's former student Oscar Bolza, is closely related to the work of Schoute and of Kerschensteiner discussed above. Sinclair considers the family

$$t^5 + xt^3 + yt + z.$$

This family is chosen such that real x, y, z exist for which the discriminant is nonzero and the number of real zeros of the polynomial is any of the possibilities $1, 3, 5$, which was not the case for Kerschensteiner's surface. In the last sentence of the Introduction, Sinclair states:

A model of the surface accompanies this investigation.

This model seems to have disappeared. Schilling reproduced Sinclair's model as Nr. 1 in the Series XXXIII. Accompanying this new model is an eight-page-long summary [Si08] which Sinclair wrote of her master's thesis. One finds Schilling's reproduction of Sinclair's model in, for example, the Mathematics Department of the Martin Luther Universität of Halle-Wittenberg; see `http://did.mathematik.uni-halle.de/modell/modell.php?Nr=Dj-001`.

In 2003, the American sculptor Helaman Ferguson made a stone model [Fe03] based on Sinclair's thesis.

Roderich Hartenstein did his *Staatsexamen* with Klein in Göttingen in 1905/06, which indicates that he probably became a high school teacher. His thesis work on the discriminant surface corresponding to

$$t^4 + xt^2 + yt + z$$

was not only recalled by Klein in his celebrated text [Kl32], but two of the string models in Schilling's Series XXXIII were based on it, and as a supplement to these models, Schilling in 1909 published a text by Hartenstein [Ha09] summarizing his thesis in 1909. This summary is referred to in 1910 in the *Bulletin of the American Mathematical Society* [BAMS10, p. 503] and also in a paper of Arnold Emch in 1935 [Em35].

To a large extent, the results of Hartenstein coincide with what Schoute already wrote concerning the polynomials of degree 4. Hartenstein discusses one novel issue: given two real bounds L, B with $L < B$, how does one find geometrically, using the discriminant surface, all polynomials $t^4 + xt^2 + yt + z$ having a zero τ such that $L \leq \tau \leq B$. One of the two Hartenstein models in Series XXXIII shows the discriminant surface corresponding to these biquadratic polynomials. The other model illustrates his solution to the problem with the two bounds.

Hartenstein writes that he already created his models in 1905/06, inspired by Klein. From footnotes on p. 10 and p. 12 one can see that he is well aware of the fact that in the meantime Klein used his results in his lectures and in the original German edition of [Kl32], which appeared in 1908. Hartenstein also gives credit to Schoute and to Kerschensteiner (cf. [Ha09, footnote on p. 15]).

His two models in Schilling's series XXXIII can be found, e.g., at `http://did.mathematik.uni-halle.de/modell/modell.php?Nr=Dj-002` and `Nr=Dj-003` (at the Martin Luther University in Halle-Wittenberg, Germany).

References

[BAMS10] *New Publications*, Bull. Amer. Math. Soc. **16** (1910), 502-506. MR1558954

[Dy93] Walther von Dyck, Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente. München : C. Wolf & Sohn, 1892 and 1893. (The original catalog published in 1892 consists of 430 pages; a supplement (Nachtrag) which appeared in 1893 contains another 135 pages.) See also `http://libsysdigi.library.uiuc.edu/ilharvest/MathModels/0007KATA/`

[Em35] Arnold Emch, *New Models for the Solution of Quadratic and Cubic Equations,* National Mathematics Magazine, **9**, No. 5 (1935), 162–164. MR1569206

[Fe03] Helaman Ferguson, *A parametric form of the quintic discriminant, very remarkable and deserves to be better known*. See `http://www.helasculpt.com/gallery/quniticmaryemI/`

[GL08] Judy Green and Jeanne LaDuke, Pioneering Women in American Mathematics: The Pre-1940 PhD's. (History of Mathematics, Vol. 34) Providence: AMS, 2009. See also the supplementary material: `http://www.ams.org/bookpages/hmath-34/PioneeringWomen.pdf` MR2464022 (2010a:01008)

[Ha09] Roderich Hartenstein, Die Diskriminantenfläche der Gleichung 4ten Grades. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M. Schilling. Neue Folge. Nr. 8.) Leipzig: Schilling, 1909. (19 pages.)

[Ke92] G. Kerschensteiner, *Geometrische Darstellung der Discriminanten dritten und vierten Grades*, pp. 168–173 in [Dy93], 1892.

[Ke93] G. Kerschensteiner, *Geometrische Darstellung der Discriminante der Hauptgleichung fünften Grades*, pp. 23–25 in the Nachtrag of [Dy93], 1893.

[Kl92] F. Klein, *Geometrisches zur Abz'"ahlung der reellen Wurzeln algebraischer Gleichungen*, pp. 3–15 in [Dy93], 1892.

[Kl32] F. Klein, Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis. New York: The Macmillan Company, 1932. MR2098410

[KNAW] Verslagen der Zittingen van de Wis- en Natuurkundige Afdeeling der Koninklijke Akademie van Wetenschappen, van 27 Mei 1893 tot 21 April 1894. Amsterdam: Johannes Müller, 1894. See also the second half of `http://www.archive.org/stream/verslagenderzit00netgoog`

[MBCZ] V. Malthe Bruun and C. Crone, *Quatre modèles représentant des surfaces développables, avec des renseignements sur la construction des modèles sur les singularités qu'ils représentent (avec quelques remarques sur les surfaces développables et sur l'utilité des modèles par M. le Docteur H.G. Zeuthen).* Paris: J. Baudry, 1877.

[S11] Catalog mathematischer Modelle für den höheren mathematischen Unterricht. Siebente Auflage. Leipzig: Martin Schilling, 1911. See also `http://libsysdigi.library.uiuc.edu/ilharvest/MathModels/0006CATA/`

[Sch93] P.H. Schoute, *Drei Fadenmodelle von entwickelbaren Flächen, die mit algebraischen Gleichungen höheren Grades in Verbindung stehen*, pp. 25–28 in the Nachtrag of [Dy93], 1893.

[Sch] Schoute biography, on `http://www-history.mcs.st-and.ac.uk/Biographies/Schoute.html`

[Si03] Mary Emily Sinclair, Concerning the Discriminantal Surface for the Quintic in the Normal Form: $u^5+10xu^3+5yu+z=0$. Master's thesis, Faculties of Arts, Literature, and Science, University of Chicago, 1903.

[Si08] Mary Emily Sinclair, Discriminantal Surface for the Quintic in the Normal Form $u^5+10xu^3+5yu+z=0$. Halle: Verlag von Martin Schilling, 8 pages, 1908.

[Sy64] J.J. Sylvester, *Algebraical Researches, Containing a Disquisition on Newton's Rule for the Discovery of Imaginary Roots, and an Allied Rule Applicable to a Particular Class of Equations, Together with a Complete Invariantive Determination of the Character of the Roots of the General Equation of the Fifth Degree, &c*, Philosophical Transactions of the Royal Society of London, Series A, **154** (1864), 579-666.

[We95] H. Weber, Lehrbuch der Algebra. Vol. 1. Braunschweig: Friedrich Vieweg und Sohn, 1895. See also `http://www.archive.org/details/lehrbuchderalgeb01weberich`

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