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Aspects of Brownian motion, by Roger Mansuy and Marc Yor, Universitext, Springer-Verlag, Berlin, 2008, xiv+195 pp., ISBN 978-3-540-22347-4

Mark Yor is one of the world's leading authorities on Brownian motion. This book, edited with his student Roger Mansuy, is a reprint of earlier lecture notes that he delivered in 1992 at ETH Zürich. Reviewing it gives an opportunity to publicize how magnificent a mathematical object Brownian motion is and how great is its intersection with other fields of analysis.

To describe Brownian motion to a mathematician who has never studied it, one may begin with a different stochastic process, called a simple random walk. One can think of this as a collection of real-valued functions on $[0, \infty)$ that are zero at zero and are parallel to the x-axis, except at the integers greater than zero, at which they either increase one unit or decrease one unit with equal likelihood. (Toss a fair coin. If it comes up heads the function goes up, if tails, down.) By $x = m$ there are 2^m possible functions, or as probabilists say, paths, and they are all equally likely to occur. Let S_m denote these paths. Obviously, $-m \leq S_m \leq m$, although for m very large $S_m = \pm m$ is very unlikely. In fact a fundamental result in probability theory, the Central Limit Theorem, states that as $m \to \infty$, $S_m/\sqrt{m}$ has the probability distribution of a normal random variable with mean zero and variance one. Therefore, in generalizing this process so that the jumps occur at $x = k/n$, for some large integer n, and all k in the positive integers including 0, we essentially must take the jump size to be $1/\sqrt{n}$ if we do not want the resulting processes to either be immediately unbounded or identically zero, as n goes to infinity. For each n denote the generalized process by X_n. We can take $\lim_{n\to\infty} X_n$, in a suitable topology, so that it is a collection of paths that are continuous on R_+. The limit process is Brownian motion, which we denote by $B = \{B_t, t \in R_+\}$. (To be more precise, we should write this as $B = \{B_t(\omega), \omega \in \Omega, t \in R_+\}$, where Ω is a probability space, since B is a collection of functions.)

Heuristically, it is easy to see that $B_0 = 0$ and, moreover, that B has stationary independent increments. By stationary we mean that all increments of the same length have the same probability distribution. Furthermore, since B is the limit of processes that increase or decrease $1/\sqrt{n}$ in an interval of length n, the paths in B have a modulus of continuity on the order of $\sqrt{t}$. The existence of such a large collection of continuous nondifferentiable functions seemed quite remarkable when Wiener first showed this in 1923.

For a measurable set $A \in R^1$ consider $\mu_t(A) := \int_0^t 1_{\{B(s)\in A\}}\, ds$, the amount of time the Brownian motion path, starting from 0, spends in A up to time t. Remarkably, for fixed t, $\mu_t(A)$ is absolutely continuous with respect to Lebesgue measure. We denote its Radon Nikodym derivative by ℓ_t^x. Of course this is a random variable since it depends on the paths in B. We form the stochastic process, i.e., collection of random variables, $\ell := \{\ell_t^x, (x,t) \in R^1 \times R_+\}$. This can be done in such a way that ℓ is continuous on $R^1 \times R_+$, or "jointly continuous", as people like to say. This stochastic process ℓ is called the local times of Brownian motion.

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There is another family of processes intimately related to Brownian motion and its local times, the δ-dimensional squared Bessel processes, $\delta \geq 0$. These are difficult to describe simply except when δ is an integer greater than or equal to 1. A k-dimensional squared Bessel process is $\{\sum_{j=1}^{k}(B_t^2)_j, t \in R_+\}$, where $(B_t)_j, j = 1, \ldots, k$, are independent Brownian motions.

Much of this book deals with the interplay between Brownian motion, its local times, and Bessel processes. The majority of the results are distributional, obtained by evaluating Laplace and Fourier transforms. Amazing connections are discovered. They are true, because they are proved. Yet more often than not they are mysteries because they do not have probabilistic explanations, i.e., quasi-physical interpretations based on the nature of the processes involved. The authors lament this fact repeatedly. This subject still awaits an overview that explains the myriad of fascinating relationships that exist.

Another intriguing aspect of Brownian motion illuminated in this book is its interplay with areas of analysis that, at first glance, seem distant from probability. This is pointed out in the chapter-by-chapter review of the book that follows.

As mentioned in the first paragraph of this review, this volume is a reprint of the first eleven chapters of Mark Yor's lecture notes: *Some aspects of Brownian motion, Part I: Some special functionals* (Lectures in Mathematics ETH Zürich, Birkhäuser-Verlag, Basel, 1992) and *Some aspects of Brownian motion, Part II: Some recent martingale problems* (Lectures in Mathematics ETH Zürich, Birkhäuser-Verlag, Basel, 1997). The remaining chapters of Part II were republished in *Random times and enlargements of filtrations in a Brownian setting* (Lecture Notes in Mathematics, 1873, Springer-Verlag, Berlin, 2006). In the Introduction the authors state that "we modified quite minimally the old eleven first chapters, essentially by completing the Bibliography." A review of *Some aspects of Brownian Motion, Part II: Some recent martingale problems*, by Richard F. Bass can be found in *Metrica* 49 (1999), 164–165. Chapters 10 and 11, the final chapters of this volume also appear in the volume reviewed by Bass.

What follows is a synopsis of the chapters, focusing on relationships amongst Brownian motion, its local times and Bessel processes, and its interplay with aspects of analysis which, on first inspection, do not appear to be related to probability.

Chapter 1 examines certain Gaussian subspaces of the Gaussian space generated by a one-dimensional Brownian motion. This leads to a consideration of the Brownian bridge and the relationship of Brownian motion and Hardy's inequality in L^2. The Fourier transform of the measure defined by $\int \lambda_{\omega,t}(dx) f(x) = \int_0^t ds f(B_s(\omega))$, where B is Brownian motion, is in L^2 almost surely. Therefore, $\lambda_{\omega,t}(dx)$ is absolutely continuous. Its family of densities ℓ_t^x are the local times of B up to time t and satisfy the occupation times formula $\int_0^t f(B_s)\,ds = \int f(y)\ell_t^y\,dy$. The deep relationship between Brownian motion and its local times is one of the most exciting aspects of the theory of Brownian motion and this book.

Chapter 2 contains several formulas for Fourier transforms and Laplace transforms of Brownian quadratic functionals that give several variations of Lévy's formula for the stochastic area of Brownian motion. It also introduces the squared Bessel process. Let $(B_t, t \geq 0)$ be Brownian motion in R^n, i.e., each component is an independent one-dimensional Brownian motion. $X_t := |B_t|^2$ is the n-dimensional squared Bessel process starting at zero. Writing X_t as a stochastic integral, one sees that this definition can be extended to δ-dimensional squared Bessel processes,

$\delta \geq 0$, starting at $x \in R^1$. We denote these processes by $(Q_x^\delta, x \geq 0, \delta \geq 0)$ and observe that for all $\delta, \delta', x, x' \geq 0$, $Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'}$, where $*$ denotes the convolution of the two probabilities. This relationship is particularly interesting when $\delta' = 0$.

The zeroth order squared Bessel process describes the total local time of Brownian motion up to a certain stopping time. This observation, one of the classical Ray–Knight Theorems, is the first result in Chapter 3. There are many variations of these isomorphism theorems involving local times of Brownian motion with different stopping rules and squared Bessel processes of different dimensions. What is obscured in concentrating on Brownian motion, but is evident when studying this property for much more general Markov processes (as in the Dynkin Isomorphism Theorem), is that there is a general relationship between the local times of a transient symmetric Markov process, and the squares of independent Gaussian processes that are related to the Markov processes.

The Ciesielski–Taylor identities, relating different properties of Bessel processes were published in 1962, one year earlier than the first Ray–Knight Theorems. Chapter 4 continues to develop the relationship between squared Bessel processes and Brownian local times to explain and extend these identities.

Chapters 5 and 7 consider windings of planar Brownian motion. Let $Z_t = X_t + iY_t$, $t \geq 0$, $Z_0 = z_0 \neq 0$, where $X_t - X_0$ and $Y_t - Y_0$ are independent Brownian motions. Let $(\theta_t(\omega), t \geq 0)$ denote a continuous determination of the argument of $(Z_u(\omega), u \leq t)$ around 0. In Chapter 5, Spitzer's classical result, $\lim_{t\to\infty} 2\theta_t/\log t \overset{law}{=} C_1$, is derived and generalized. (Here C_1 is a Cauchy random variable, i.e., it has probability density function $1/\pi(1+x^2)$, $-\infty < x < \infty$.) This problem is approached through a consideration of winding numbers of "Brownian lace", which is a complex Brownian bridge. In Chapter 7 Spitzer's result is extended to a multidimensional result for the winding numbers $(\theta_t^1, \ldots, \theta_t^n)$ of planar Brownian motion around n-points. In a further extension, Brownian motion in R^3 is considered and asymptotic laws are obtained for its winding numbers around a finite number of oriented straight lines and certain unbounded curves. The linking number of two closed curves in R^3, which do not intersect each other, was defined by Gauss. This is considered, with modifications, for two Brownian curves. Modifications are necessary since two independent Brownian motions in R^3 almost surely intersect each other.

Chapter 6 studies the the integral over intervals of time of the exponential of Brownian motion with drift. Such expressions occur in Asian options in mathematical finance. This chapter follows Chapter 5 because of the similarities in some of the techniques used in the proofs. Results obtained are extended to more general Markov processes and Lévy processes in particular.

In a 1939 paper Lévy shows that both $\int_0^1 ds 1_{(B_s>0)}$ and $\sup\{t < 1 : B_t = 0\}$ have an arc-sine distribution. This result has been extended in many directions. In Chapter 8 it is extended in three directions, in which $(B_s \geq 0)$ is replaced by (i) a symmetrized Bessel process with dimension $0 < \delta < 2$; (ii) a Walsh Brownian motion; and (iii) a singularly perturbed reflecting Brownian motion, that is $(|B_s| - \mu\ell_s^0, s \geq 0)$ where $(\ell_s^0, s \geq 0)$ is the local time of $(B_s \geq 0)$ at 0 and $\mu > 0$. It is remarkable that $(\ell_s^0, s \geq 0)$ has a critical role in all three of these extensions.

Chapter 9 continues the study of $X \equiv (|B_s| - \mu\ell_s^0, s \geq 0)$. Another Ray–Knight Theorem is obtained, this time relating the joint distribution of $(\ell_{\tau_s^\mu}^x(X); x \geq 0)$ and $(\ell_{\tau_s^\mu}^{-x}(X); x \geq 0)$ to squares of Bessel processes, where $(\tau_s^\mu, s \geq 0)$ is the inverse of the local time $(\ell_s^\mu, s \geq 0)$ at μ for the process X. Several consequences of this theorem are investigated.

The local time $(\ell_t^x, (x,t) \in R \times R_+)$ of Brownian motion has a continuous version. In fact it is Hölder continuous in both x and t. Therefore, one can take the Hilbert transform of ℓ_t^x at a point $x \in R$. In Chapter 10 this is used to define and obtain the distribution of $\lim_{\epsilon\to 0} \int_0^t (1_{(|B_s - a|) \geq \epsilon)}/(B_s - a))\, ds$ and related random variables. Some applications are given including a study of excursions of Bessel processes.

The final chapter, Chapter 11 has the intriguing title, "Probabilistic representations of the Riemann zeta function and some generalizations related to Bessel processes". The authors immediately state that "the aim of this chapter is not to discuss Riemann's hypothesis!, but... to present some of the... realtions between... (the) zeta function... and Brownian motion". They show that the zeta function can be expressed in terms of first hitting time of 1, by the sum of two three dimensional Bessel processes starting from zero, and then they go on to obtain other fascinating equalities.

I agree with Bass's summation in his review of Chapters 10–18 of these notes, mentioned above. "This book... is a delight. I was reminded of Ramanujan's notebooks; here too there are a huge number of beautiful and unexpected relationships between seemingly disparate objects." The reader will marvel at the authors' knowledge and expertise. Nevertheless, the book makes clear that although the mathematical study of Brownian motion is almost one hundred years old, the directions for continued study and new investigations remain unlimited.

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