# BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 3, July 2011, Pages 471–473
S 0273-0979(2011)01328-3
Article electronically published on January 11, 2011

*The geometry of infinite-dimensional groups*, by Boris A. Khesin and Robert Wendt, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 51, Springer-Verlag, Berlin, 2009, xii+304 pp., ISBN 978-3-540-77262-0, e-ISBN 978-3-540-77263-7

Khesin and Wendt's book can be thought of as a link in a long chain of works devoted to the development of mathematical structures that model physical phenomena and the study of the properties of such structures. We might consider calculus, which Newton developed to model mechanics, as the beginning of this chain. We then proceed to Euler and Lagrange to fluid mechanics and then on to Hamilton and Jacobi, who developed more advanced techniques to solve the classical equations, and on to Liouville and Poincaré and then Kolmogorov. Note that the mathematics of this entire chain was almost entirely analysis, which is understandable since the primordial concept of the whole chain was the derivative.

From the late twentieth century until today, however, there has been a clear shift from analysis through geometry and currently primarily to algebra. The analysis is mostly long established and remains in the background without needing to be discussed. The foreground is a study of symmetries and therefore group actions or representations, and from this to topological methods followed by algebraic geometry, which now plays a leading role in mathematical physics.

This evolution is analogous to that of art over the same time period. Beginning with the strictly representational, one goes through impressionist and expressionist periods to the nonobjective and even abstract. There is in most cases an object to portray, but the portrayal itself is at best in the background and is superseded by what one might call artistic effects—patterns, harmonies of color, etc.

The book itself starts with (possibly infinite-dimensional) Lie groups and their algebras, defines the adjoint and co-adjoint representations, and then proceeds to central extensions, thus introducing the most basic of the algebraic machinery. It goes on to define Hamiltonian systems on the cotangent bundle of a Lie group. If the Hamiltonian function is left (or right) invariant, one can reduce the system to an equation on the dual of the Lie algebra. This the authors call the "Euler equation" and indeed it is the Euler equation in the familiar examples of the motion of a rigid body or of an incompressible inviscid fluid. The chapter concludes with a presentation of the moment map (also known in the literature as the "momentum

2010 *Mathematics Subject Classification.* Primary 22-02; Secondary 22F05, 22F30.

map") and its use in symplectic reduction, which reduces the Hamiltonian system to one of smaller dimension.

The second chapter is devoted to examples. They are all groups of invertible self-maps of some manifold or central extensions of such groups. These include loop groups, the Bott-Virasoro group, the group of all diffeomorphisms of a manifold, and those subgroups of it which preserve some structure on the manifold. These examples are followed by the group of symbols of pseudo-differential operators on functions on the circle. Both the Bott-Virasoro group and the groups of pseudo-differential symbols are used to construct completely integrable systems—the former yields the Korteweg-de Vries equation and the latter the Kadomtsev-Petviashvili.

The third chapter is about various moduli spaces and related objects such as the Chern-Simons functional. Groups of gauge transformation, which are also infinite-dimensional Lie groups, act on various function spaces, and the moduli spaces can be construed as spaces of orbits of the actions. Said spaces frequently have a complex or even algebraic structure, and they may have singularities. In this chapter (among other places) there is a prominent theme of real versus complex analogy. One has the orientation of a real manifold versus a meromorphic $n$-form, a flat connection versus a holomorphic bundle, affine versus elliptic Lie algebras, singular homology versus polar homology, etc.

Finally there are ten appendices devoted to different applications of the material in the chapters and to additional related constructions. They include root systems in semi-simple Lie algebras, certain Kähler geometries, and the classical optimal mass transport problem which goes back to Monge.

All in all, the book presents a very large variety of topics and thus emphasizes breadth rather than depth. Many things are discussed rather briefly and frequently proofs are not included. However, there are ample references to the enormous bibliography, which contains 393 listings, so the interested reader can easily delve further if he or she wishes. The book may be most useful as a way to get an overview of the subject (actually a collection of subjects) or as a window through which to glimpse any one of the subjects and then go deeper using the references provided.

In book reviews for the *Bulletin* it is customary to describe the field in which a book lies, rather than the book itself. With the present book, this is difficult because it is not within a field; it rather blankets a field or even several fields. Most related topics are touched on in the book itself.

One can, however, identify a central construct, the diffeomorphism group of a manifold. It, together with subgroups and extensions, plays a major role both in the book and in related works. The Bott-Virosaro group is a one-dimensional central extension of the diffeomorphism group of the circle. More generally $\mathcal{D}(\Omega)$, the identity component of the diffeomorphism group of a compact manifold, possibly with boundary, (e.g., a bounded domain in $\mathbf{R^n}$) serves as the configuration space for the motion of a fluid which fills $\Omega$. $\eta(t)$, a curve in this group, can describe a fluid motion where $\eta(t)(x)$ gives the position at time $t$ of that fluid particle which was originally at $x$. The fluid motion will depend on a Riemannian metric $\langle\, ,\, \rangle$ on $\Omega$ and on its corresponding volume element $\mu$. Then the kinetic energy will be $K(\eta, \dot{\eta}) = \int_\Omega \langle \dot{\eta}(t)(x), \dot{\eta}(t)(x)\rangle \mu(x)$, where "$\dot{}$" means time derivative. If the fluid has negligible viscosity, the only internal force on it will be due to the compressing caused by the motion, and this can be expressed as a potential energy $V(\eta) = \int_\Omega f(\frac{\eta^*(\mu)}{\mu})\mu$, where $f$ is some function defined by the physics of the fluid. Then a fluid motion would be a stationary curve of the Lagrangian $\int_0^T K(\dot\eta(t), \eta(t)) - V(\eta(t))dt$. It could equivalently be the flow of the Hamiltonian $H := K + V$.

If the fluid is incompressible, the configuration space is limited to $\mathcal{D}_\mu(\Omega) := \{\eta \in D(\Omega) | \eta^*(\mu) = \mu\}$, which is sometimes called the "volumorphism group". It is a Lie subgroup of $\mathcal{D}(\Omega)$. Motion of an inviscid incompressible fluid is a stationary curve of $\int_0^T K(\eta(t), \dot\eta(t))\, dt$ for $\eta(t)$ restricted to the subgroup $\mathcal{D}_\mu(\Omega)$. Such a curve is a geodesic, so properties such as its stability are determined by the curvature of $\mathcal{D}_\mu(\Omega)$.

$\mathcal{D}(\Omega)$ acts on $\mathcal{V}$, the set of volume elements on $\Omega$, by pullback; that is, $A : \mathcal{D}(\Omega) \times \mathcal{V} \to \mathcal{V}$ by $A(\eta, \mu) = \eta^*(\mu)$. Each set of volume elements with a given total volume is a single orbit of $\mathcal{D}(\Omega)$, which of course can be identified with $\mathcal{D}(\Omega)/\mathcal{D}_\mu(\Omega)$. Given two points in an orbit, the shortest curve in $\mathcal{D}(\Omega)$ which joins them is deemed the optimal mass transport, as explained in Appendix 5 of the book.

$\mathcal{D}(\Omega)$ also acts by pullback on $\mathcal{M}$, the space of Riemannian metrics on $\Omega$. In this case $\mathcal{M}/\mathcal{D}(\Omega)$ is infinite dimensional and is usually called the space of Riemannian structures on $\Omega$. It could also be thought of as a moduli space for Riemannisn metrics. It has singularities only at orbits for which the group of isometries of the metric is nontrivial. These isometry groups are finite-dimensional compact Lie subgroups of $\mathcal{D}(\Omega)$. The Ricci flow $g(t)$ is of course a curve in the space of metrics and it naturally descends to $\mathcal{M}/\mathcal{D}(\Omega)$. The descended curve is conceptually the one to work with when one deals with metrics modulo diffeomorphisms, as in [1].

If $\Omega$ admits a symplectic form $\omega$, one can also consider the subgroup $\mathcal{D}_\omega(\Omega) = \{\eta \in \mathcal{D}(\Omega) | \eta^*(\omega) = \omega\}$, which is known as the "symplectomorphism group". It is also an infinite-dimensional subgroup of $\mathcal{D}(\Omega)$ and one can study its geodesics as with $\mathcal{D}_\mu(\Omega)$. Furthermore, if $\mathcal{S}ym$ is the set of all symplectic forms on $\Omega$, one again gets an action $A : \mathcal{D}(\Omega) \times \mathcal{S}ym \to \mathcal{S}ym$ defined by pullback. Again, one can consider the orbit space $\mathcal{S}ym/\mathcal{D}(\Omega)$. It is easy to see that each orbit is an open subset of a single deRham cohomology class and thus is finite dimensional, but it is not always known how many orbits are in a class.

From all this we see that the diffeomorphism group plays a pervasive role in mechanics, geometry, and topology, and it thus appears in many places in the book and elsewhere.

## References

[1] R. S. Hamilton *A compactness property for solutions of the Ricci flow.* Amer. J. Math. **117** (1995), no. 3, 545–572. MR1333936 (96c:53056)

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