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*Words: notes on verbal width in groups*, by Dan Segal, London Mathematical Society Lecture Note Series, Vol. 361, Cambridge University Press, Cambridge, 2009, xii+121 pp., ISBN 978-0-521-74766-0

Words are basic objects of group theory, appearing naturally from the very beginning of group theory. A *group word* in letters $x_1, \ldots, x_n$ is a product $y_1 y_2 \cdots y_m$, where $y_i \in \{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}\}$.

A word $w$ in letters (variables) $x_1, \ldots, x_n$ can be naturally regarded for every group $G$ as a function from $G^n$ to $G$. It is natural to identify words with such functions (so that, for instance, the word $x^{-1}xy$ is equal to the word $y$). It means that we identify group words in letters $x_1, \ldots, x_n$ with elements of the free group generated by them.

We say that a word $w$ in letters $x_1, \ldots, x_n$ is a *law* (or an *identity*) in a group $G$ if $w$ becomes equal to the trivial element of $G$ for any choice of the values in $G$ of the variables $x_i$, i.e., if the corresponding function $G^n \to G$ is constantly equal to the trivial element of the group.

For example, $a^{-1}b^{-1}ab$ is a law $G$ if and only if $G$ is commutative. Other examples of classes of groups defined by laws they satisfy are nilpotent groups, solvable groups, and Burnside groups. Study of group laws and *varieties* of groups defined by them is already a classical subject in group theory; see for instance the monograph [6].

For any group $G$ and any word $w$ there exists the biggest quotient of $G$ (i.e., quotient by the smallest subgroup) for which $w$ is a law. Namely, it is the quotient by the subgroup $w(G)$ generated by the values of the word $w$ for different assignments of the values of its variables in $G$. It is easy to see that $w(G)$ is always a normal (even characteristic) subgroup of $G$. The subgroups of the form $w(G)$ are called *verbal*, and are important objects in the study of group laws. The most classical and ubiquitous example is the *derived subgroup* $G'$ generated by the values of the commutator word $a^{-1}b^{-1}ab$. Other important examples are the subgroups of the derived and lower central series of a group.

If $F_n$ is the free $n$-generated group, then $F_n/w(F_n)$ is the free group in the variety of groups for which $w$ is a law (i.e., an $n$-generated group belongs to the variety if and only if it is a quotient of $F_n/w(F_n)$).

Many classical problems of group theory are centered around the topics of laws and verbal subgroups. For instance, the famous Burnside problem asks when the group $F_n/w(F_n)$ is infinite for the word $w = x^m$; see [1].

Every element of the verbal group $w(G)$ is a product of some values of the function $w$. A natural question to ask is, What is the smallest number $n$ such that every element of $w(G)$ can be represented as a product of at most $n$ values of $w$? If such a number $n$ exists, then the word $w$ is said to be of *finite width* in $G$. If $G$ is a finite group, then obviously every word $w$ has a finite width. However, we may ask whether the width of $w$ is bounded in some class of finite groups (e.g., in all finite groups, in all finite simple groups, in all finite $p$-groups, etc.) In fact a

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natural question is existence of a function $f$ such that the width of $w$ in every finite group $G$ (belonging to some class) is bounded by $f(d)$, where $d$ is minimal size of a generating set of $G$.

In general (i.e., not only for finite groups) a group $G$ is said to be *$w$-elliptic*, if $w$ has finite width in $G$. A group $G$ is said to be *verbally elliptic* if it is $w$-elliptic for every word $w$ (this terminology was introduced by P. Hall). If $\mathcal{C}$ is a class of finite groups, then a word $w$ is *uniformly elliptic* in $\mathcal{C}$ if there exists a function $f(d)$ such that the width of $w$ in $G \in \mathcal{C}$ is not more than $f(d)$, where $d$ is the smallest number of generators of $G$.

An example of results on ellipticity is a theorem of V. Romankov stating that every finitely generated virtually nilpotent group (i.e., a group containing a nilpotent subgroup of finite index) is verbally elliptic. A similar result belongs to P. Stroud, who showed that every finitely generated abelian-by-nilpotent group is verbally elliptic. On the other hand, free non-abelian groups $F_k$ of rank $k \geq 2$ are never $w$-elliptic (except for the "silly cases" of trivial words and words of the form $x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k} g$, where $\gcd(e_1, \ldots, e_k) = 1$ and $g \in [F_k, F_k]$); see Chapter 3 of *Words* (the book under review).

Questions related to width of words in classes of finite groups are especially important for the theory of *profinite groups*. A topological group is called profinite if it is compact and has a basis of neighborhoods of the trivial element consisting of subgroups. Equivalently, a profinite group is an inverse limit of finite groups.

Study of profinite groups is in some sense equivalent to the study of the set of finite groups approximating it (i.e., of the set of quotients by open normal subgroups). Therefore, different *uniform* estimates (e.g., on width of verbal subgroups) for finite groups are used to prove results about properties of profinite groups.

For example, in a profinite group $G$ the verbal subgroup $w(G)$ is *closed* if and only if the word $w$ has finite width in $G$ (an observation attributed to Brian Hartley by [8]). A word $w$ has finite width in a profinite group $G$ if and only if there exists a uniform upper bound on the width of $w$ in every finite continuous quotient of $G$.

If $\mathcal{C}$ is a formation of finite groups (i.e., a class closed under taking quotients and finite subdirect products), then a word $w$ is uniformly elliptic in $\mathcal{C}$ if and only if $w(G)$ is closed in $G$ for every pro-$\mathcal{C}$ group; see [8, Proposition 4.1.3].

Relation between width and topology in profinite groups was used around 1975 by J.-P. Serre (see [9, §42, Exercise 6]) to prove that in a finitely generated pro-$p$ group every subgroup of finite index is open. This implies that the class of open subgroups coincides with the class of subgroups of finite index, so that topology of the group is defined in purely algebraic terms, and all homomorphisms between such groups are continuous. Serre's proof is based on the fact that in a nilpotent group generated by $d$ elements, every element of the commutator subgroup $[G, G]$ is a product of at most $d$ commutators, i.e., that the commutator word $a^{-1}b^{-1}ab$ is uniformly elliptic in the formation of finite nilpotent groups (which contains the formation of finite $p$-groups).

Note that the condition for the group $G$ to be finitely generated cannot be dropped. For instance, if $G = C_p^{\mathbb{N}}$ is the infinite Cartesian power of a cyclic $p$-group, then there exist $2^{\mathfrak{c}}$ many homomorphisms $G \to C_p$, hence uncountably many subgroups of index $p$. But there are only countably many open subgroups of index $p$.

Classification of finite simple groups opens new methods of studying verbal subgroups of finite groups. Many interesting results in theory of verbal subgroups of finite groups were obtained using the classification, i.e., reducing them to questions about finite simple groups and examining the corresponding series of alternating and matrix groups (which is usually still rather complicated and involves difficult combinatorics, algebraic geometry, number theory, etc.)

After a dormant period (results of V. Romankov are from the early 1980s, work of P. Hall and P. Stroud is from 1960s), the subject of verbal width in finite groups became very active in the recent years, with many amazing results, most of which use the classification of finite simple groups. For instance, A. Shalev [10] proved that for any nontrivial group word $w$, every element of every sufficiently large finite simple group is a product of three values of $w$; see also [3]. In particular, this shows that every nontrivial word is uniformly elliptic in the class of all finite simple groups. Another result in this direction is the proof of Ore's Conjecture: every element of every non-abelian finite simple group is a commutator; see [4]. Note that for any nontrivial group word $w$ and a simple group $G$, either $w(G) = 1$ (i.e., $G$ satisfies the law $w = 1$) or $w(G) = G$. For instance, every element of a non-abelian simple group is a product of commutators.

One of the important recent results coming from the study of width of words in finite groups is the positive answer (by D. Segal and N. Nikolov [7]) to the question whether the natural generalization of Serre's theorem is true:

**Theorem 1.** *A subgroup of finite index of a finitely generated profinite group is open.*

Their proof is based on the following result (see [8, Theorem 4.2.1] and [7]):

**Theorem 2.** *Let $d \in \mathbb{N}$ and a word $w$ be such that the group $F_d/w(F_d)$ is finite (where $F_d$ is the free $d$-generated group). Then there exists $f = f(w, d)$ such that $w$ has width at most $f$ in every $d$-generated finite group.*

Let us show how Theorem 2 implies Theorem 1. Let $G$ be a profinite group topologically generated by $d$ elements, and let $N$ be a subgroup of finite index in $G$. Then taking all conjugates of $N$ and intersecting them, we get a subgroup $N_1 \leq N$ that is a normal subgroup of finite index in $G$. It is enough to show that $N_1$ is open. One then finds a word $w$ such that $F_d/w(F_d)$ is finite and $w(G/N_1) = 1$. Then $w(G) \leq N_1$ and $w(G)$ is closed, since $w$ has finite width. The group $F_d/w(F_d)$ is finite, hence the quotient of $G$ by $\overline{w(G)} = w(G)$ is finite too (and is a quotient of $F_d/w(F_d)$), which implies that $w(G)$, $N_1$, and $N$ are open.

Theorem 1 is probably one of the most amazing results that was proved using classification of finite simple groups.

Dan Segal's book *Words* is a short but comprehensive overview of the techniques and results in verbal width of finite and profinite groups. It is very well written and pleasant to read.

It starts with a general discussion of words and verbal subgroups. Chapter 2 discusses the results of P. Stroud, K. George, and V. A. Romankov on verbal ellipticity of different classes of groups (usually related to nilpotency: finitely generated virtually nilpotent, finitely generated virtually abelian-by-nilpotent, virtually soluble minimax, etc.)

The main part of *Words* is Chapter 4. In particular, it gives an introduction to the work of D. Segal and N. Nikolov on Serre's problem.

One of central results described in Chapter 4 is the following remarkable theorem of A. Jaikin describing all uniformly elliptic words in the class of all finite $p$-groups (more generally, all finite nilpotent $\pi$-groups, where $\pi$ is a set of primes).

**Theorem 3.** *We say that a word $w$ in $k$ variables, seen as an element of the free group $F$ of rank $k$, is a $J(p)$-word if $w \notin F''(F')^p$. We say, for a set $\pi$ of primes, that $w$ is a $J(\pi)$-word, if it is a $J(p)$-word for every $p \in \pi$. Then a nontrivial word $w$ is uniformly elliptic in the class of finite nilpotent $\pi$-groups if and only if it is a $J(\pi)$-word.*

A word $w$ is called *uniformly elliptic* if it is uniformly elliptic in the class of all finite groups. A word $w$ is uniformly elliptic if and only if it has finite width in every finitely generated profinite group, which in turn is equivalent to the condition that $w(G)$ is a closed subgroup for any finitely generated profinite group $G$.

The problem of describing all uniformly elliptic words is still open. A. Jaikin's theorem gives a necessary condition: the word has to be a $J(p)$-word for every prime $p$. It is not known if this condition is also sufficient. Some partial results in this direction are discussed in §4.8 of *Words*.

The last chapter of the book describes results on verbal ellipticity of algebraic groups (a result due to Ju. Merzlyakov [5]), $p$-adic analytic groups [2], and in some profinite groups associated with them.

Finally, the appendix contains a selection of open questions in the subject. For example, a "major challenge of the subject" is the question whether $x^q$ is uniformly elliptic in finite groups.

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