BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 3, July 2011, Pages 475–480
S 0273-0979(2011)01335-0
Article electronically published on March 9, 2011

The ergodic theory of lattice subgroups, by Alexander Gorodnik and Amos Nevo, Princeton University Press, Princeton, 2010, xiv + 121 pp., US \$29.95/US\$60.00, ISBN 978-0-691-14185-5

Given a dynamical system $T : X \to X$ (with discrete time for simplicity of notation) on a metric space, it is desirable to describe the behavior of orbits $x, T(x), T^2(x), \ldots \in X$. While this is usually very difficult for individual points, the study of typical orbits is often significantly easier. To describe what we mean by "a typical point" we need the following definitions. A probability measure μ on X is called invariant and T is called measure preserving if $\mu(T^{-1}B) = \mu(B)$ for all measurable $B \subset X$, or equivalently if $\int f \circ T \, d\mu = \int f \, d\mu$ for every integrable f. Given an invariant probability measure for $T : X \to X$ we have three completely general facts from the early days of ergodic theory:

- Poincaré recurrence states that μ-a.e. point x returns to any neighbhorhood U of x; in fact there are infinitely many integers $n_1 < n_2 < \cdots < n_k < \cdots$ with $T^{n_k}(x) \in U$ for all k. This is the "pigeon hole principle" of ergodic theory.
- If $f \in L^2(X, \mu)$, then von Neumann's ergodic theorem states that the time average

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n(x)) \tag{1}$$

  converges in L^2 as $N \to \infty$.
- Birkhoff's pointwise ergodic theorem states that for any μ-integrable f the time average in (1) converges for μ-a.e. x.

By using that the interval $[1, N]$ is "almost invariant by translation", one quickly sees that the limit of (1) must be a T-invariant function. The invariant probability measure μ (or T) is called ergodic (with respect to μ) if any measurable invariant set has measure 0 or 1, or equivalently if any invariant measurable function is constant μ-a.e. This is the "irreducibility assumption" in ergodic theory. If now μ is ergodic, the above implies the time averages in (1) actually converge to the space average $\int f \, d\mu$. We note that Poincaré recurrence and the L^2-ergodic theorem are very easy and quick to prove. However, the pointwise ergodic theorem requires ideas similar to the proof of the Lebesgue density theorem in measure theory; cf. [14] and the survey [12] for similarities.

An important extension of the above setup is to allow actions $G \times X \to X$ of locally compact metrizable groups G. The recurrence extends to any action of a noncompact group and the L^2-ergodic theorem extends easily to actions of "amenable groups". (A group G is called amenable if there exist so-called Følner sets in G, which are—similar to intervals in $\mathbb{Z}$ or $\mathbb{R}$—almost invariant under translation.) While pointwise ergodic theorems for special classes of amenable groups were known for some time (see [12] for the history of the developments), the pointwise ergodic theorem for general amenable groups was only proved in 1999 by E. Lindenstrauss [9].

2010 *Mathematics Subject Classification.* Primary 22D40; Secondary 22E30, 28D99, 11D45.

Before we discuss other classes of groups, let us define a few more properties of measure preserving transformations and actions.

- A measure preserving action of G on a probability space (X, μ) is ergodic if any $f \in L^2(X, \mu)$ with $f \circ g = f$ for all $g \in G$ is necessarily constant a.e.
- An invariant probability measure μ for a transformation[1] $T : X \to X$ is weak mixing if the only eigenfunction $f \in L^2$ of the isometry $f \to f \circ T$ on L^2 are the constant functions.
- A measure preserving action of G on a probability space (X, μ) is mixing if for any two $f_1, f_2 \in L^2(X, \mu)$ we have
$$\int (f_1 \circ g) f_2 \, \mathrm{d}\mu \to \int f_1 \, \mathrm{d}\mu \int f_2 \, \mathrm{d}\mu$$
as $g \to \infty$ (i.e., as g leaves any compact subset of G).
- A measure preserving action of G on a probability space (X, μ) is mixing of all orders if for any $k \geq 1$, any measurable sets $A_0, \ldots, A_k$ we have
$$\mu(A_0 \cap g_1 A_1 \cap \cdots \cap g_k A_k) \to \mu(A_0) \cdots \mu(A_k)$$
as $g_j \to \infty$ and $g_j^{-1} g_\ell \to \infty$ for all $j, \ell = 1, \ldots, k$ with $j \neq \ell$.

In the case of $G = \mathbb{Z}$ it is easy to see that mixing implies weak mixing which in turn implies ergodicity. The simple example of a circle rotation $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $R_\alpha(x) = x + \alpha \mod \mathbb{Z}$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ shows that ergodicity does not imply weak mixing. There are also examples of weak mixing systems which are not mixing, but these are more difficult to construct. Somewhat surprisingly all known examples of mixing transformations (or actions of $\mathbb{Z}$) are mixing of all orders. Whether this is just a coincidence or whether indeed mixing implies mixing of all orders is the oldest open problem in ergodic theory, going back to Rokhlin and Halmos (from about 1950). For $G = \mathbb{Z}^2$, we already know that mixing does not imply mixing of all orders due to the example of Ledrappier [8].

In many ways the ergodic theory of general amenable groups is similar to the ergodic theory of single transformations and is highly developed; see [13]. For noncompact semisimple Lie groups, which are one of the main interests of the book by Gorodnik and Nevo, many new phenomena arise, which often more than compensate for the lack of amenability. While the ergodic theory of semisimple Lie groups has also a long tradition and has become very important, e.g., for application to number theory, before the work of Gorodnik and Nevo there was no satisfying theory of general ergodic theorems of such actions.

We would like to discuss some of the surprises and strengths of the study of measure preserving actions of semisimple Lie groups on probability spaces. By a theorem of Howe and Moore [7] (which is known as "decay of matrix coefficients at infinity"), we have that any ergodic action of a noncompact simple[2] connected Lie group is automatically mixing, which in particular implies that ergodicity of the action of G implies ergodicity of the induced action of any unbounded subgroup $H < G$. Furthermore, Mozes [11] has shown that an ergodic action of a noncompact simple Lie group is automatically mixing of all orders.

[1] For the definition of weak mixing for an action one needs to consider finite-dimensional invariant subspace of $L^2(X, \mu)$, see [1].

[2] For ease of exposition we assume here and in the following that G is simple (e.g., $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 2$) instead of just semisimplicity (e.g., $G = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ for $n \geq 2$).

To see an example of how the theorem of Howe and Moore can become relevant in applications, let us discuss counting problems on homogeneous varieties. Here the most basic case is the Gauss circle problem: What is the number $N(R)$ of integer points $\mathbf{n} \in \mathbb{Z}^2$ of norm $\|\mathbf{n}\| < R$? Gauss showed that by considering unit squares centered at the integer points, one can quickly get the estimate $N(R) = \pi R^2 + O(R)$, where the $O(R)$ comes from the area of an annulus around the circle of radius R. While the error term is conjectured to be $O_\epsilon(R^{\frac{1}{2}+\epsilon})$, one can at least obtain a small improvement over the $O(R)$ by an equidistribution statement: The circle of radius R considered modulo $\mathbb{Z}^2$ as a subset of $(\mathbb{R}/\mathbb{Z})^2$ equidistributes with respect to the Lebesgue measure. Using this, one can show that the following cases mostly cancel each other: the cases where some part of the square belongs to the circle while the lattice point at the center does not and, respectively, the cases where the lattice point at the center belongs to the circle but some part of the square does not. This can be used to improve the error term to the form $N(R) = \pi R^2 + O(R^{1-\delta})$ for some small $\delta > 0$.

In the case of a similar hyperbolic counting problem, the consideration of an equidistribution problem is significantly more important. Let us recall that $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ is the upper half plane model of the hyperbolic plane. We endow $\mathbb{H}$ with the usual Riemannian metric defined by $\frac{(\mathrm{d}x)^2+(\mathrm{d}y)^2}{y^2}$ and recall that $\mathrm{SL}_2(\mathbb{R})$ acts via the isometries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \text{ for } z \in \mathbb{H}.$$

Then one important case of a hyperbolic counting problem consists in finding an asymptotic formula for

$$N(R) = \left|\{\gamma \cdot i : \gamma \in \mathrm{SL}_2(\mathbb{Z}), d(i, \gamma i) < R\}\right|,$$

which was, for instance, considered by Selberg. Motivated by the Gauss circle problem, one could again try to replace each point in the orbit $\mathrm{SL}_2(\mathbb{Z}) \cdot i$ by a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ and proceed along similar lines. However, a simple straightforward generalization of the above argument is bound to fail in at least two ways. First the fundamental domain is unbounded, and second the area of the hyperbolic ball grows exponentially with the radius, which makes the area of an annulus of fixed width comparable in size to the area of the ball it encloses. Therefore, a simple generalization of the above counting argument would at best produce an error term of the same order of magnitude as the main term. However, using mixing of the left action of $\mathrm{SL}_2(\mathbb{R})$ on the finite measure space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$, one can show the equidistribution of the images of large circles in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Margulis used in his thesis [10] such an equidistribution statement to prove as a particular case the asymptotic formula

$$N(R) = \frac{1}{2\operatorname{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})} \operatorname{vol}(B_R(i)) + o(\operatorname{vol}(B_R(i))). \tag{2}$$

For much more general symmetric varieties Duke, Rudnick, and Sarnak [3] have obtained, by using harmonic analysis on homogeneous spaces, similar asymptotic formulas for the number of integer points on the variety. Just shortly afterwards, Eskin and McMullen [4] were able to prove under slightly weaker assumptions the same results by using the above mixing property. This started a completely new and by now very rich chapter in the interaction between ergodic theory and number theory; see, e.g., [5] and the much more recent [6].

One can also improve on the error term above. However, for this, another feature of noncompact simple Lie groups in the form of a spectral gap is needed. We say a unitary representation π of G on a Hilbert space H has a spectral gap if there exist some $\epsilon > 0$ and some compact set $K = K^{-1} \subset G$ such that the convolution operator

$$\chi_K(v) = \frac{1}{m_G(K)} \int_K \pi(g)v \,\mathrm{d}m_G$$

satisfies

$$\|\chi_K v\|_H \leq (1-\epsilon)\|v\|_H \text{ for } v \in \left(H^G\right)^{\perp}, \tag{3}$$

where H^G is the subspace of vectors that are fixed under G. A group G has Kazdhan's property (T) if there exist such an ϵ and K (independent of π and H) so that (3) holds for all unitary representations (e.g., $\mathrm{SL}_2(\mathbb{R})$ does not have property (T) while $\mathrm{SL}_3(\mathbb{R})$ has property (T)). However, even for simple Lie groups G that do not have property (T), the spectral gap is still a very persistent property: Every irreducible representation of a simple Lie group G has a spectral gap. Furthermore, if $\Gamma < G$ is a lattice so that G/Γ has finite volume, then the natural unitary representation of G on $L^2(G/\Gamma)$ also has a spectral gap.

Spectral gap is highly desirable because of the following facts. By a theorem of Cowling, Haagerup, and Howe [2], a representation with a spectral gap allows an error rate in the decay of matrix coefficients at infinity. To state this theorem in a concrete case, let $f_1, f_2 \in C_c^\infty(X)$ with $X = G/\Gamma$ being a quotient of a connected simple linear group G by a lattice $\Gamma < G$. Then we have

$$\left| \int (f_1 \circ g) f_2 \,\mathrm{d}m_X - \int f_1 \,\mathrm{d}m_X \int f_2 \,\mathrm{d}m_X \right| \leq C_{f_1,f_2} \|g\|^{-\delta},$$

where m_X is the Haar measure on X, C_{f_1,f_2} is a constant depending on the functions f_1, f_2, $\|g\|$ is the norm of the matrix g, and $\delta > 0$ only depends on the spectral gap. One application of this theorem is that it allows one to prove a more concrete form of the error term in the above mentioned counting problems.

Let us indicate, in a rather crude way, a connection between spectral gap and ergodic theorems. For this suppose that the measure preserving action of G on a probability space (X, μ) is ergodic with spectral gap. By assumption, the constant functions are the only functions in L^2 on which G acts trivially. Therefore, if we iterate the convolution operator χ_K in definition (3) of spectral gap for a given $f \in L^2(X, \mu)$, we obtain

$$\left\| \chi_K^n(f) - \int f \,\mathrm{d}\mu \right\|_2 \leq (1-\epsilon)^n \|f\|_2.$$

Here χ_K^n is the convolution operator defined by the nth convolutional power $\frac{1}{m_G(K)^n} 1_K^{*n}$ of the normalized characteristic function of K. Note that 1_K^{*n} has support contained in K^n. Hence, if we ignore the precise shape of 1_K^{*n}, then $\chi_K^n(f)$ resembles an ergodic average over the set K^n and so the above looks like an L^2-ergodic theorem with a very good error rate. Obtaining ergodic theorems where the average is taken just over large natural sets (without the need of introducing somewhat arbitrary density functions as above) is a harder problem, which is precisely the topic of the book.

In the book *Ergodic theory of lattice subgroups* by Gorodnik and Nevo, the authors obtain very general ergodic theorems for semisimple Lie groups and their lattices. In fact, the authors allow general products (S-algebraic groups) of semisimple algebraic groups defined over $\mathbb{R}$, $\mathbb{Q}_p$, or (with some restrictions) even $\mathbb{F}_p((t))$. In such a group G or any of its lattices Γ certain natural classes of sets G_t (with $t \in \mathbb{R}_+$ or $t \in \mathbb{N}_+$)—for instance norm balls—can be used to define ergodic averages. Under fairly mild assumptions on G_t the authors give the L^2 ergodic theorem and the pointwise ergodic theorems for any ergodic action of G or a lattice Γ. Furthermore, if the action has a spectral gap, very good error rates for the pointwise ergodic theorem are proved (without assuming any smoothness property of the function). Due to the omnipresence of spectral gap for semisimple groups, this result stands in stark contrast to the ergodic theory of single transformations or even amenable groups, where such a result is not true. The authors also apply their methods to the problem of estimating the number of lattice points, i.e. $N(t) = |\Gamma \cap G_t|$; e.g., in the counting problem (2) and in higher-dimensional analogues, more general sets other than balls are considered, and for these an error term is proved which is close to the one obtained previously for balls by Selberg, and Lax and Phillips.

To summarize, the book will certainly become the standard reference for ergodic theorems of semisimple Lie groups and their lattices and will help to stimulate further interactions between ergodic theory and number theory.

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