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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
WILLIAM H. MEEKS III AND JOAQUÍN PÉREZ

MR1906593 (2003c:53013) 53A10
Hutchings, Michael; Morgan, Frank; Ritoré, Manuel; Ros, Antonio
Proof of the double bubble conjecture.
Annals of Mathematics. Second Series **155** (2002), *no.* 2, 459–489.

About 2300 years ago, Zenodorus made the first known attempt to show that the circle is the shortest curve in the plane enclosing a given area. Weierstrass developed the analysis needed to provide a complete proof in the nineteenth century. The proof of the optimality of the round sphere in three-dimensional space for enclosing a given volume was completed in 1882 by Schwarz [H. A. Schwarz, Göttingen Nachr. **1884**, 1–13; JFM 16.0232.04]. One motivation for these investigations was to develop a theory of surfaces that would explain the configurations achieved by objects such as soap films and soap bubbles. Soap bubble experiments, starting with the pioneering investigations of Plateau [J. Plateau, *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires*, Gauthier-Villars, Paris, 1873; JFM 06.0516.03; C. V. Boys, *Soap bubbles*, third edition, Dover, New York, 1959], give rise not only to spheres, but also to complicated froths of bubbles. When exactly two connected components are enclosed, the shape assumed by a soap froth is known as a "standard double bubble". It is made of three pieces of round spheres, meeting along a common circle at an angle of 120 degrees. In the case of two equal volumes, a standard double bubble consists of two spherical pieces of equal radius, separated by a flat disk, all meeting along a common circle at an angle of 120 degrees. The double bubble conjecture asserts that the standard double bubbles are the most efficient possible shapes to use in enclosing any two given volumes in $\mathbf{R}^3$. More precisely, if v_1 and v_2 are two specified positive numbers, the conjecture states that a standard double bubble has smallest possible area among all the surfaces enclosing and separating volumes v_1 and v_2. For the case of equal volume regions, where $v_1 = v_2$, the conjecture was solved in [J. Hass, M. Hutchings and R. Schlafly, Electron. Res. Announc. Amer. Math. Soc. **1** (1995), no. 3, 98–102 (electronic); MR1369639 (97b:53014)]. Also previously established was the lower-dimensional problem of finding a shortest curve enclosing any two areas in the plane. The solution there is a planar double bubble, three circular arcs meeting at two points at 120 degree angles [J. Foisy et al., Pacific J. Math. **159** (1993), no. 1, 47–59; MR1211384 (94b:53019)]. In this paper Hutchings, Morgan, Ritoré and Ros complete the story in 3-space, establishing the double bubble conjecture in $\mathbf{R}^3$ for all pairs of volumes v_1, v_2.

The study of bubbles and surfaces that minimize area while enclosing given volumes fits into a general class of problems regarding optimal geometric configurations. These problems bring together techniques from geometry, analysis and computation. The existence and regularity results related to the double bubble problem are due to F. J. Almgren, Jr. [Mem. Amer. Math. Soc. **4** (1976), no. 165, viii+199 pp.; MR0420406 (54 #8420)] and J. E. Taylor [Ann. of Math. (2) **103**

(1976), no. 3, 489–539; MR0428181 (55 #1208a)]. Their work established the existence and regularity of an optimal bubble-like surface enclosing a prescribed set of two or more volumes. Such a surface consists of finitely many smooth pieces, each having constant mean curvature. The pieces meet along a singular locus consisting of curves where three surfaces meet at 120 degrees, and finitely many points where six surfaces meet. The latter type of singularity does not arise in double bubble problems, but might in problems involving more regions, one reason they are more difficult. This type of singular surface was physically observed by Plateau in the mid 1800's in his study of soap bubbles. It is a triumph of geometric measure theory, a mixture of analytic and geometric techniques developed to handle this kind of problem, that a mathematical framework of area minimizing surfaces was created that appears to accurately model physically observed soap films. See [F. Morgan, *Geometric measure theory*, Third edition, Academic Press, San Diego, CA, 2000; MR1775760 (2001j:49001)] for geometric and analytic background in this area.

Conglomerations of constant mean curvature surfaces in 3-space are much too general to classify completely. However, work of White, Foisy and Hutchings using symmetry techniques showed that an optimal double bubble must be a surface of revolution [see M. Hutchings, J. Geom. Anal. **7** (1997), no. 2, 285–304; MR1646776 (99j:53010)]. In other words the solution to our problem must be symmetric under rotation around an axis. This implies that the pieces of a minimizing surface are subsurfaces of a well-studied class of constant mean curvature surfaces, the Delaunay surfaces.

A puzzling aspect of the double-bubble story is that a chief difficulty in its solution is the problem of establishing connectedness of the regions. It is conceivable that the most efficient surface enclosing two volumes actually encloses three (or more) connected regions in space. If this really occurred, then the best shape for a canteen holding a certain amount of milk and a certain amount of lemonade (without mixing of course) would enclose two separate regions containing milk, as well as one or more containing lemonade. Such intuitively unlikely possibilities are difficult to eliminate. Hutchings studied the possible configurations and, while not establishing connectedness in all cases, was able to show that there are a limited number of possible configurations for a minimizer. So, for example, one of the enclosing volumes is always connected.

In the paper under review, Hutchings, Morgan, Ritoré and Ros begin by further limiting the possible shapes that a minimizer enclosing two volumes might take. In particular they show that there are at most three connected components enclosed and separated by such a surface. The volumes of two of the components add to v_1, and the third component has volume v_2. They then analyze the possible geometric configurations of the remaining surfaces. For each possible nonstandard minimizer, they construct a deformation that preserves the volume enclosed in each of the two regions, but decreases the total surface area. In other words, they show that a nonstandard configuration cannot be stable; it cannot be optimal even when compared only to nearby shapes. The deformations are constructed by rotating portions of the bubble around a second axis, perpendicular to the axis of revolution. The rotated portions are carefully chosen to balance and to blend smoothly into the rest of the bubble. In the nonstandard cases, this results in deformations that cause cancelling increases and decreases in the volumes enclosed by each region, while decreasing the area of the enclosing surface. All rivals are eliminated by appropriate constructions of such deformations, leaving only the standard double bubble as a possible minimizer.

While this work has closed the story of the double-bubble problem in 3-space, many intriguing and beautiful geometric problems have been revealed along the way, including many still open. We remark for example that the solution of the double bubble conjecture has recently been extended to $\mathbf{R}^4$ by B. Reichardt et al. [“Proof of the double bubble conjecture in $\mathbf{R}^4$”, Pacific J. Math., to appear].

From MathSciNet, March 2011

Joel Hass

MR2128712 (2006a:53007) 53A10; 49Q05, 53C42

Meeks, William H., III; Pérez, Joaquín; Ros, Antonio

The geometry of minimal surfaces of finite genus. I. Curvature estimates and quasiperiodicity.

Journal of Differential Geometry **66** (2004), *no.* 1, 1–45.

Let $\mathcal{M}$ be the space of properly embedded minimal surfaces in $\mathbb{R}^3$ with genus zero and two limit ends, normalized so that every surface $M \in \mathcal{M}$ has horizontal limit tangent plane at infinity and the vertical component of its flux equals one. In this article, the authors prove that if a sequence $\{M(i)\}_i \subset \mathcal{M}$ has the horizontal part of the flux bounded from above, then the Gaussian curvature of the sequence is uniformly bounded.

This curvature estimate yields compactness results. The important application of the curvature estimates is to obtain a theorem in this paper which is a descriptive theorem on the geometry of a properly embedded minimal surface M with finite genus and two limit ends. In particular, such a surface M has bounded curvature and is conformally a compact Riemann surface punctured in a countable closed subset with two limit points; the spacing between consecutive ends is bounded from below in terms of the lower bound of the Gaussian curvature; M is quasiperiodic in the sense that there exists a divergent sequence $V(n) \in \mathbb{R}^3$ such that the translated surfaces $M + V(n)$ converge to a properly embedded minimal surface of genus zero, two limit ends, a horizontal limit tangent plane at infinity and with the same flux as M.

In Part II [Invent. Math. **158** (2004), no. 2, 323–341; MR2096796], the authors prove that a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus cannot have one limit end. In Part III [W. H. Meeks, III, J. Pérez and A. Ros, “The geometry of minimal surfaces of finite genus. III. Bounds on the topology and index of classical minimal surfaces”, preprint; per bibl.], results in Part II and this paper will be applied to obtain a bound on the number of ends and on the index of stability for a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and at least two ends. This bound depends only on the genus of the surface.

An important theoretic tool in deriving the curvature bound is a “blowing-up process on the scale of topology” of a sequence of properly embedded minimal surfaces. This blow-up argument produces a new sequence of properly embedded minimal surfaces in $\mathbb{R}^3$ which is uniformly locally simply-connected. The proof depends heavily on the local and global compactness and regularity theorems by Colding and Minicozzi. Another important tool in the Colding-Minicozzi theory is the 1-sided curvature estimate.

From MathSciNet, March 2011

Fei-Tsen Liang

MR2096796 (2006a:53008) 53A10; 49Q05, 53C42

Meeks, William H., III; Pérez, Joaquín; Ros, Antonio

The geometry of minimal surfaces of finite genus. II. Nonexistence of one limit end examples.

Inventiones Mathematicae **158** (2004), *no.* 2, 323–341.

This manuscript is the second in a series of papers whose goal is to describe the topology, geometry, asymptotic behavior and conformal structure of properly embedded minimal surfaces M in $\mathbb{R}^3$ with finite genus. In this manuscript, the authors prove that a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus cannot have one limit end.

The proof of this result relies on a series of deep papers by Colding and Minicozzi on the structure of embedded minimal planar domains in $\mathbb{R}^3$. Their results are used to prove that for any properly embedded minimal surface M in $\mathbb{R}^3$ with finite genus and one limit end, any sequence of homothetic scalings $M_n = \lambda_n M$, $\lambda_n \to 0$, has a subsequence converging C^α, $0 < \alpha < 1$, to a limit minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ whose leaves are smooth outside the origin. This regularity result implies that the curvature of M decays quadratically in terms of the radial function, which in turn is used to prove that such a surface cannot exist.

The main theorem in [P. Collin et al., "The geometry, conformal structure and topology of minimal surfaces with infinite topology", J. Differential Geom., to appear] states that a limit end of a properly embedded minimal surface M with horizontal limit tangent plane must be a top or bottom end and hence M can have at most two limit ends. By the results of Collin [Ann. of Math. (2) **145** (1997), no. 1, 1–31; MR1432035 (98d:53010)] and W. H. Meeks, III and H. Rosenberg [Ann. of Math. (2) **161** (2005), no. 2, 727–758], the asymptotic behavior of an M with finite topology can be characterized as follows: each annular end of M is asymptotic to an end of a plane, catenoid or helicoid, with the helicoid-type end occurring only in the case M has one end and is not a plane. Furthermore, M is conformally diffeomorphic to a compact Riemann surface $\overline{M}$ punctured in a finite number of points and M can be defined analytically in terms of meromorphic data on $\overline{M}$. If M has finite genus and two limit ends, then results [Part I, W. H. Meeks, III, J. Pérez and A. Ros, J. Differential Geom. **66** (2004), no. 1, 1–45; MR2128712] show that M has bounded curvature and that M is conformally a compact Riemann surface punctured in a countable closed subset with two limit points. In Part III [W. H. Meeks, III, J. Pérez and A. Ros, "The geometry of minimal surfaces of finite genus. III. Bounds on the topology and index of classical minimal surfaces", preprint; per bibl.], these results and the new results in this paper will be applied to obtain a bound on the number of ends and on the index of stability for a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and at least two ends. This bound depends only on the genus of the surface.

REVISED (November, 2005)

From MathSciNet, March 2011

Fei-Tsen Liang

MR2123932 (2006e:53012) 53A10; 53C42

Colding, Tobias H; Minicozzi, William P., II

The space of embedded minimal surfaces of fixed genus in a 3-manifold. III. Planar domains.

Annals of Mathematics. Second Series **160** (2004), *no.* 2, 523–572.

This paper is the third in a series where the authors describe the spaces of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. In [Part I, Ann. of Math. (2) **160** (2004), no. 1, 27–68; MR2119717 (2006a:53004)], [Part II, Ann. of Math. (2) **160** (2004), no. 1, 69–92; MR2119718 (2006a:53005)] and [Part IV, Ann. of Math. (2) **160** (2004), no. 2, 573–615; MR2123933], the authors described the case where the surfaces are topologically disks on any fixed small scale. This paper considers general planar domains and proves a result (Corollary III.3.5) which is needed in [T. H. Colding and W. P. Minicozzi, II, op. cit.; MR2123933].

There are two local models for embedded minimal disks. One model is the plane and the other is a piece of helicoid. In the first four papers of this series, the authors show that every embedded minimal disk is either a graph of a function or is a double spiral staircase where each staircase is a multi-valued graph. This will be done by showing that if the curvature is large at some point (and hence the surface is not a graph), then it is a double spiral staircase. To prove that such a disk is a double spiral staircase, the authors first prove that it can be decomposed into N-valued graphs where N is a fixed number. Corollary III.3.5 asserts that in an embedded minimal disk, above and below any given multi-valued graph there are points of large curvature and thus, by [T. H. Colding and W. P. Minicozzi, II, op. cit.; MR2119717 (2006a:53004); op. cit.; MR2119718 (2006a:53005)], there are other multi-valued graphs both above and below the given one. Iterating this gives the decomposition of such a disk into multi-valued graphs. Part IV [T. H. Colding and W. P. Minicozzi, II, op. cit.; MR2123933] will deal with how the multi-valued graph fits together and, in particular, prove regularity of the set of points of large curvature—the axis of the double spiral staircase.

There are two main themes in this paper. The first is that stability leads to improved curvature estimates. This allows one to find large graphical regions. These graphical regions lead to two possibilities: either they "close up" to form a graph, or a multi-valued graph forms. The second theme is that in certain important cases one can rule out the formation of multi-valued graphs. The technique developed here applies both to general planar domains and to certain topological annuli in an embedded minimal disk; the latter is used in [T. H. Colding and W. P. Minicozzi, II, op. cit.; MR2123933].

Planar domains arise when one studies convergence of embedded minimal surfaces of a fixed genus in a fixed 3-manifold. This is due to (1) and (2) of Theorem 0.1 which asserts that any sequence of embedded minimal surfaces of fixed genus has a subsequence which consists of uniformly planar domains away from finitely many points. Theorem 0.5 and Corollary 0.7 describe a neighborhood of each of the finitely many points where the genus concentrates.

To describe general planar domains in [T. H. Colding and W. P. Minicozzi, "The space of embedded minimal surfaces of fixed genus in a 3-manifold. V. Fixed genus", in preparation], we need in addition to the results of [T. H. Colding and W. P. Minicozzi, II, op. cit.; MR2119717 (2006a:53004); op. cit.; MR2119718 (2006a:53005);

op. cit.; MR2123933], a key estimate for embedded stable annuli which is the main result of this paper, namely Theorem 0.3. This estimate asserts that such an annulus is graphical away from its boundary if it has only one interior boundary component and if this component lies in a small (extrinsic) ball. Theorem 0.3 is a kind of effective removable singularity theorem. The stability condition in Theorem 0.3 is used in two ways: to obtain a pointwise curvature bound and to show that certain sectors have small curvature. Combining Theorem 0.3 and the solution of a Plateau problem of Meeks and Yau yields Corollary 0.4, which provides us with a stable graphical annulus separating the boundary components of an embedded minimal annulus. In [T. H. Colding and W. P. Minicozzi, op. cit., in preparation], where the authors give the actual "pair of pants" decomposition, Corollary 0.4 separates each pair of pieces in the decomposition by putting in minimal graphical annuli in the complement of the domains.

From MathSciNet, March 2011

Fei-Tsen Liang

MR2123933 (2006e:53013) 53A10; 53C42

Colding, Tobias H.; Minicozzi, William P., II

The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected.

Annals of Mathematics. Second Series **160** (2004), *no.* 2, 573–615.

This paper is the fourth in a series where the authors describe the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. The key is to understand the structure of an embedded minimal disk in a ball in $\mathbb{R}^3$. This was undertaken in [Part I, T. H. Colding and W. P. Minicozzi, II, Ann. of Math. (2) **160** (2004), no. 1, 27–68; MR2119717 (2006a:53004); Part II, Ann. of Math. (2) **160** (2004), no. 1, 69–92; MR2119718 (2006a:53005)] and the global version of it will be completed in this paper.

The first four papers of this series show that every embedded minimal disk is either a graph of a function or a double spiral staircase where each staircase is a multiply-valued graph. This is done by showing that if the curvature is large at some point (and hence the surface is not a graph), then it is a double spiral staircase like the helicoid. To prove that such a disk is a double spiral staircase, the authors showed in the first three papers that it is built out of N-valued graphs where N is a fixed number. In this paper the authors deal with how the multivalued graphs fit together and, in particular, prove regularity of the set of points of large curvature—the axis of the double spiral staircase.

The main results are Theorem 0.1 (the lamination theorem) and Theorem 0.2 (the one-sided curvature estimate). Theorem 0.1 is stated in the global case where the lamination is, in fact, a foliation. An immediate consequence of Theorem 0.1 is that if an embedded minimal disk starts to spiral very tightly, then it can unwind only very slowly; that is, in a whole extrinsic tubular neighborhood it continues to spiral tightly and fills up almost the entire space. Thus Theorem 0.1 yields the global version of the statement that every embedded minimal disk, if not a graph, is a double spiral staircase with a singular set $\mathcal{S}$ which is a Lipschitz curve.

Theorem 0.2 is referred to as the one-sided curvature estimate since it gives a curvature estimate for embedded minimal disks Σ on one side of the domain with $\partial\Sigma \subset \partial B_{2r_0}$; Theorem 0.2 asserts that, for some $\varepsilon > 0$, the components of $B_{r_0} \cap \Sigma$ intersecting $B_{\varepsilon r_0}$ are graphs. Rescaling the catenoid shows that the simply-connectedness of Σ is crucial in Theorem 0.2. Corollary 0.4 is an almost immediate consequence of Theorem 0.2, which asserts that two sufficiently close components of an embedded minimal disk must each be a graph.

Theorem 0.2 is proved by contradiction. Suppose that Σ is an embedded minimal disk in the halfspace $\{x_3 > 0\}$ and has low points with large curvature. Starting with such a point, Corollary III.1.3 decomposes Σ into disjoint multi-valued graphs using the existence of nearby points with large curvature (given by Proposition I.0.11), a blow-up argument and [T. H. Colding and W. P. Minicozzi, II, op. cit.; MR2119717 (2006a:53004); op. cit.; MR2119718 (2006a:53005)]. The key point is then to show that we can in fact find such a decomposition where the "next" multi-valued graph starts off a definite amount below where the previous multi-valued graph started off. In fact, this definite amount is shown to be a fixed fraction of the distance between where the two graphs started off. Iterating this eventually forces Σ to have points where $x_3 < 0$, which is the desired contradiction.

Since the authors announced their results, a number of interesting theorems have been proved using Theorem 0.1 and Theorem 0.2. For instance, in [Duke Math. J. **107** (2001), no. 2, 421–426; MR1823052 (2002a:53008)], using Theorem 0.2, the authors give an alternative proof of the so-called generalized Nitsche conjecture originally proved by P. Collin by very different arguments. In [Part V, "The space of embedded minimal surfaces of fixed genus in a 3-manifold. V. Fixed genus", in preparation], using Theorem 0.2 and [Part III, Ann. of Math. (2) **160** (2004), no. 2, 523–572; MR2123932], the authors prove that any embedded minimal annulus in a ball and with a small neck can be decomposed by a simple closed geodesic into two graphical sub-annuli; moreover, they give a sharp bound for the length of this closed geodesic in terms of the separation between the graphical sub-annuli. The results in this paper also play a key role in the proof of Calabi-Yau conjectures for embedded surfaces in [T. H. Colding and W. P. Minicozzi, II, "The Calabi-Yau conjectures for embedded surfaces", preprint, `arxiv.org/abs/math.DG/0404197`], the main result of which was a chord-arc bound for possibly non-compact embedded minimal disks, relating the extrinsic and intrinsic distances. Thus [T. H. Colding and W. P. Minicozzi, II, op. cit.; `arxiv.org/abs/math.DG/0404197`] provides us with intrinsic versions of all of the results of this paper.

Using Theorems 0.1 and 0.2, Meeks and Rosenberg proved that the plane and helicoid are the only complete properly embedded simply-connected minimal surfaces in $\mathbb{R}^3$. Using this, W. H. Meeks, III [Duke Math. J. **123** (2004), no. 2, 329–334; MR2066941 (2005d:53014)] proved that the singular set $\mathcal{S}$ in Theorem 0.1 is a straight line perpendicular to the foliation.

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From MathSciNet, March 2011

Fei-Tsen Liang

MR2480608 (2010d:53011) 53A10; 49Q05

Weber, Matthias; Hoffman, David; Wolf, Michael

An embedded genus-one helicoid.

Annals of Mathematics. Second Series **169** (2009), *no.* 2, 347–448.

In [H. Karcher, F. S. Wei and D. A. Hoffman, in *Global analysis in modern mathematics (Orono, ME, 1991; Waltham, MA, 1992)*, 119–170, Publish or Perish, Houston, TX, 1993; MR1278754 (95k:53011)] a surface in the Euclidean 3-space satisfying the following properties was constructed:

(1) it is a properly immersed minimal surface;
(2) it has genus one and one end asymptotic to the helicoid;
(3) it contains a single vertical line and a single horizontal line.

A surface with the above properties is called a genus-one helicoid. Note that embeddedness is not a property required of a genus-one helicoid. Although computer-generated images and computational estimates showed beyond reasonable doubt that this genus-one helicoid was embedded, there is no noncomputational proof.

In the present article the authors prove the existence of an embedded genus-one helicoid. They point out that they believe that this embedded surface is actually the genus-one helicoid of Hoffman, Karcher and Wei, and conjecture that there is a unique embedded genus-one helicoid.

The surface presented in the paper is the first example (except for the helicoid) of a properly embedded minimal surface in $\mathbb{R}^3$ with finite topology and infinite total curvature.

The surface is exhibited as a geometric limit of periodic embedded minimal surfaces starting at the singly periodic genus-one helicoid [D. A. Hoffman, H. Karcher and F. S. Wei, Comment. Math. Helv. **74** (1999), no. 2, 248–279; MR1691949 (2000h:53008)]. The periodic surfaces $\mathcal{H}_k$, $k \geq 1$, are invariant under a cyclic group of screw motions generated by σ_k (the rotation by $2\pi k$ about the vertical axis followed by a vertical translation of $2\pi k$). The embeddedness of the limit surface is inherited from the embeddedness of the approximating periodic surfaces. To prove it, the authors use that in this particular minimal surface setting, the condition of being embedded is both open and closed in families.

An important feature of the paper is the development of a theory of singular flat structures on the tori that admit infinite cone angles.

From MathSciNet, March 2011

Antonio Alarcón