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ABOUT THE COVER: EARLY IMAGES OF MINIMAL SURFACES

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The survey by Meeks and Pérez in this issue opens with a score of images of complete minimal surfaces in space; pictures like these powerfully and succinctly communicate the subtle richness of the subject. Drawings have accompanied mathematical descriptions of minimal surfaces almost since the subject began: Meusnier's 1785 discovery [Meu] of the helicoid and rediscovery of the catenoid (he was apparently unaware of Euler's 1744 description [Eul] of that surface in the simpler setting of a minimum among surfaces of revolution) are accompanied by illustrations (see Figures 1 and 2).

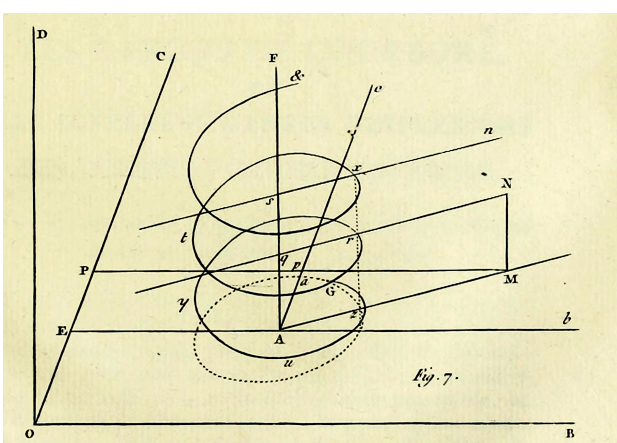

FIGURE 1. Meusnier's 1785 image of a helicoid [Meu].

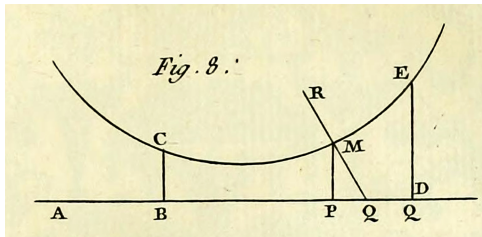

FIGURE 2. Meusnier's 1785 image of the generating curve of a catenoid [Meu].

However, this custom of illustrating geometric arguments in a research paper has come into and gone out of fashion over the centuries. Many of the nineteenth-century articles that described important new minimal surfaces relied entirely on the reader's imagination to see the surface, possibly partly as a reaction to emphasis by some eminent French mathematicians on the primacy of the power of analysis. The first images of Riemann's striking minimal surface (see [MIP, Figure 7]) seem to have appeared many decades after Riemann's (posthumously published) description in 1867 [Rie]. Schwarz [Sch] pointedly bucks this trend with some beautiful and intricate drawings (see Figure 3) of his breakthrough triply periodic surfaces (see also [MIP, Figure 6]). His student Neovius continued in that style, illustrating his 1883 thesis [Neo] with the marvelous image (see Figure 4) of a surface which extends to all of space by sending handles to the edges of every fundamental cube used to tile space.

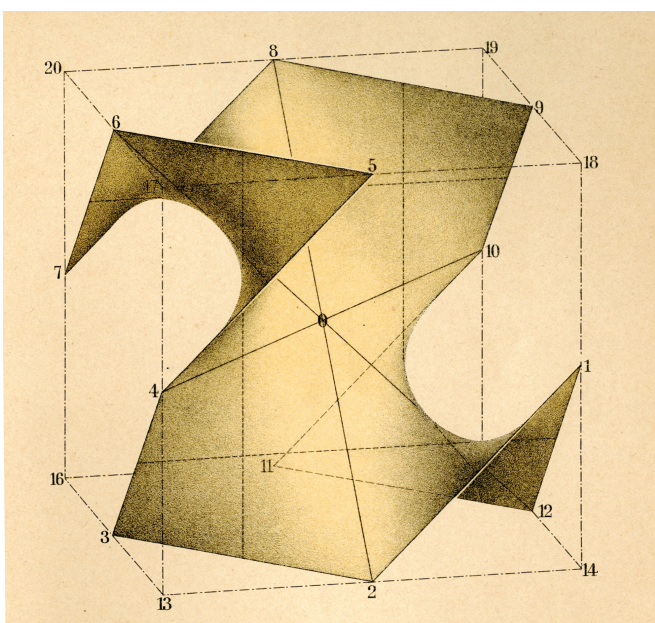

FIGURE 3. Schwarz's image of a triply periodic surface, from his collected works of 1890 [Sch].

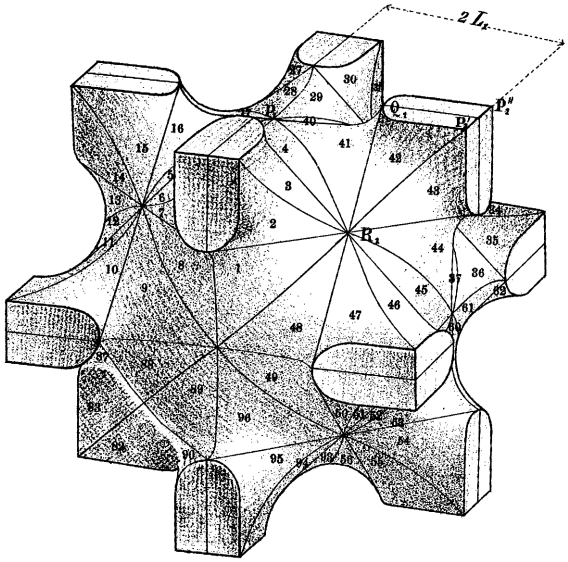

FIGURE 4. Neovius's image of a triply periodic minimal surface from his 1883 thesis [Neo].

The cover image, courtesy of David Hoffman, dates from the time of the discovery by Hoffman and Meeks [HMI] that Costa's surface [Cos] was embedded, i.e., had no self-intersections. This milestone surface was the first complete embedded minimal surface of finite topology found since the helicoid and arrived at a time of increasing evidence that no such example might exist. It was the leading edge of a flood of increasingly complicated examples, opening a new chapter in the long history of this topic, as described in the article by Meeks and Pérez [MIP] in this issue.

At that moment in the early 1980s, the role of illustrations changed: Hoffman and Meeks used the computational resources they found available to them then (including the talented programmer J. Hoffman) not only to merely create illustrations, but as a tool to investigate geometric features to help them prove theorems. In the case of Costa's surface, they were able to detect symmetries not apparent in Costa's expressions involving Weierstrass elliptic functions. Once discovered, these symmetries were straightforward to verify but key to establishing the embeddedness of the Costa surface; further elements of the argument were similarly guided by graphical experiments, now informed by the symmetries. (See [Hof] for a beautiful account of this process of discovery.) Thereafter, many new surfaces followed in quick succession: the method Hoffman, Meeks, and others (notably Karcher) pioneered in this setting was a conversation between visual aspects of the minimal surfaces and advancing theory, each supporting the other.

The picture on the cover is from just after Hoffman and Meeks had finished their proof and were seeking ways to explain to audiences the essential geometric characteristics of Costa's surface. Presently of course, the role of computational visualization to both illustrate finished theory and aid in investigation of minimal surfaces is commonplace.

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