BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 4, October 2011, Pages 507–508
S 0273-0979(2011)01346-5
Article electronically published on June 15, 2011

STANLEY'S INVARIANT THEORY SURVEY

DAVID J. BENSON

> Immediately following the commentary below, this previously published article is reprinted in its entirety: Richard P. Stanley, *Invariants of finite groups and their applications to combinatorics*, Bull. Amer. Math. Soc. (N.S.) **1** (1979) no. 3, 475–511.

When this article by Stanley first appeared, I was a graduate student at Cambridge, studying finite group theory under the supervision of John Thompson. I didn't know any commutative algebra beyond what we learned as undergraduates. But the article impressed on me the immediacy and power of commutative algebra techniques in studying objects associated with finite group actions.

It took me a while to absorb enough of the techniques of commutative algebra to be able to digest Stanley's paper properly, and the understanding I gained from this process has since permeated my own mathematics. The most obvious result was that I wrote a book on polynomial invariants of finite groups. The influence of Stanley's paper on the shape of this book is explicitly acknowledged in its introduction:

> This book is based on a lecture course I gave at Oxford in the spring of 1991. My starting point for these lectures was the excellent survey article of Richard Stanley, which I strongly recommend to anyone wishing to get an overview of the subject. The influence of this article will be apparent in almost every part of this book.

One of the most interesting aspects to me of Stanley's article was the emphasis placed on the Cohen–Macaulay, Gorenstein, and complete intersection conditions in commutative algebra. Trying to import these ideas into the cohomology of groups has led me and others to some rather startling ways in which the cohomology ring of a finite group resembles an invariant ring, as long as care is taken to apply the concepts in a derived fashion. The beginning of this process was a paper I wrote with Jon Carlson [1] in which we showed that if the cohomology ring of a finite group is Cohen–Macaulay, then it is Gorenstein. If the ring is not Cohen–Macaulay, one still gets a duality spectral sequence in local cohomology; this was formulated by Greenlees in [3]. This turns out to be a shadow at the ring theoretic level of the statement that the cohomology ring is always derived Gorenstein. The precise formulation of this was first made in a paper of Dwyer, Greenlees and Iyengar [2]; in particular, one needs more than just the ring structure of cohomology to formulate the derived Gorenstein property. The extra information amounts to giving the A_∞-structure, or equivalently the cochains on BG, as a differential graded algebra up

Received by the editors June 4, 2011.
2010 *Mathematics Subject Classification.* Primary 20C15.

©2011 American Mathematical Society

to quasi-isomorphism. One outcome of this was my regularity conjecture for finite group cohomology, recently proved by Symonds [7]; at around the same time he also proved a closely related statement for rings of polynomial invariants of finite groups [6].

The obvious next question is what it should mean for the cohomology of a finite group to be a derived complete intersection. The first indications that this is an interesting question came from Ran Levi's thesis [5], where he showed that there is a dichotomy between polynomial growth and (almost) exponential growth for $H_*(\Omega(BG^\wedge_p);k)$, the homology of the loops on the p-completion of BG. This is analogous to the dichotomy for Tor over a local ring given by Gulliksen [4]. For a suitable derived definition of complete intersection, it should be true that the cohomology of a finite group satisfies this condition if and only if $H_*(\Omega(BG^\wedge_p);k)$ has polynomial growth. This subject is still in need of clarification.

Looking at Math Reviews, it is clear that Stanley's article has influenced many other mathematicians in a wide range of subjects, including coding theory, combinatorics, algebraic topology, commutative algebra, representation theory, and even noncommutative algebra.

References

1. D. J. Benson and J. F. Carlson, *Projective resolutions and Poincaré duality complexes*, Trans. Amer. Math. Soc. **132** (1994), 447–488. MR1142778 (94f:20100)
2. W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar, *Duality in algebra and topology*, Adv. in Math. **200** (2006), 357–402. MR2200850 (2006k:55017)
3. J. P. C. Greenlees, *Commutative algebra in group cohomology*, J. Pure Appl. Algebra **98** (1995), 151–162. MR1319967 (96f:20084)
4. T. H. Gulliksen, *On the deviations of a local ring*, Math. Scand. **47** (1980), 5–20. MR600076 (82c:13022)
5. R. Levi, *On finite groups and homotopy theory*, Ph.D. thesis, University of Rochester, NY, 1993.
6. P. Symonds, *On the Castelnuovo–Mumford regularity of rings of polynomial invariants*, Ann. of Math., to appear.
7. ———, *On the Castelnuovo–Mumford regularity of the cohomology ring of a group*, J. Amer. Math. Soc. **23** (2010), 1159–1173. MR2669710

Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, Scotland