COMMENTARY ON
“AN ELEMENTARY INTRODUCTION TO THE LANGLANDS PROGRAM”
BY STEPHEN GELBART

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When I was asked to suggest a Bulletin article worthy of reprinting, I knew the answer right away: An Elementary Introduction to the Langlands Program by Steven Gelbart, published by the Bulletin in 1984, remains one of the best reviews of the Langlands Program, even though this subject has expanded tremendously in the intervening years. This paper has greatly influenced my own research and is in fact one of my all-time favorites. I still keep my old xerox copy on roughly cut cheap yellow paper, made in a Moscow library in my student years, as a memento. It is precisely the kind of review article that the Bulletin strives to publish: describing a fascinating area of mathematics in a way that is accessible to nonspecialists.

Gelbart introduces in it the key concepts of the Langlands Program appealing to very little mathematical background. He starts with a brief recollection of basic number theory and a review of some “classical themes,” such as the Local-Global Principle in number theory, modular forms, and Artin’s $L$-functions. He then moves on to the key concept of automorphic representations of reductive groups over the adèles. This sets the stage for the conjectural Langlands’ correspondence relating $n$-dimensional representations of the Galois group of a number field (or a function field) $F$ and automorphic representations of the group $GL_n$ over the ring of adèles of $F$. Introducing the Langlands dual group, Gelbart then presents Langlands’ general functoriality principle (which Langlands himself considers as the central tenet of his Program). The last chapter reviews what was known about all this at the time when Gelbart’s article went to press.

So much has happened since then! Though it was conceived initially [12] as a bridge between number theory and harmonic analysis, the Langlands Program has moved to other areas of mathematics, such as geometry, and even to quantum physics. It is tempting to think of it as a “grand unified theory” of mathematics, since it ties together so many different disciplines.

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André Weil has written (in a letter to his sister) about the deep analogy between
number theory and the geometry of complex algebraic curves, with the theory of
algebraic curves over finite fields appearing as the “go-between”. Hence Weil talked
about a “trilingual text”, in which each of the above three subjects appears as one
of the columns. Likewise, we can talk about three parallel tracks of the Langlands
Program. (For more on this, see the recent Séminaire Bourbaki [6].) The two tracks
involving algebraic curves (over finite fields and over the complex field) require
a new geometric reformulation whose essential elements are already contained in
Drinfeld’s important paper [3] published around the time when Gelbart’s review
was submitted.

The geometric Langlands Program has been developing extensively for the last 20
years and a lot of progress has been made. In particular, the geometric Langlands
correspondence has been proved for $GL_n$ in [7, 9] (generalizing earlier work [5]
[15]). For other groups a substantial part of the correspondence for curves over $\mathbb{C}$
has been proved by Beilinson and Drinfeld in [1]. This work brought ideas from
two-dimensional conformal field theory into the subject (see [4, 5] for a review).
Quantum field theories of a different kind, the four-dimensional supersymmetric
gauge theories, also turned out to be closely related. Recent work by Witten and
collaborators (starting with [10]) linking geometric Langlands correspondence to the
so-called S-duality in these theories has already brought a host of exciting new ideas
into both physics and mathematics (see [6] for an exposition and further references).
Hence we can talk about the “fourth track” of the Langlands Program—in the realm
of quantum physics.

In the meantime, many important results have also been obtained in the “clas-
sical” Langlands Program, described in Gelbart’s article. The celebrated Shimura–
Taniyama–Weil conjecture (which implies Fermat’s Last Theorem), proved by
Wiles, Taylor and others [19, 18, 2], may be viewed as a special case of the Lang-
lands correspondence for the field of rational numbers: it relates two-dimensional
Galois representations on the first étale cohomology of elliptic curves over $\mathbb{Q}$ to the
automorphic representations of the group $GL_2$ over the adèles of $\mathbb{Q}$ encoded by
certain modular forms on the upper half-plane. Similar methods and ideas have
since been applied in greater generality.

The Langlands correspondence for $GL_n$ has been proved in the case of curves
over finite fields by L. Lafforgue [11]. The recent proof of the “fundamental lemma”
by Ngô [16] was a breakthrough that has led to the stabilization of the Arthur–
Selberg trace formula and many far-reaching consequences, as one can learn from
the ambitious Book Project [17]. And there are some new ideas [13, 8, 14] on how
to prove the functoriality principle in general.

This is just a small selection of recent results. But even from this incomplete
list one can see clearly that this is a very exciting time to study the Langlands
Program.

Like a well-aged bottle of Bordeaux, Gelbart’s article still stands out as one of
the best introductions to the subject. I am honored to present it to the readers
of the Bulletin, and I hope that they will be seduced by it, and the Langlands
Program, just like I was when I first read it.
References


[3] V.G. Drinfeld, Two-dimensional \( \ell \)-adic representations of the fundamental group of a curve over a finite field and automorphic forms on \( GL(2) \), Amer. J. Math. 105 (1983) 85–114. MR0692107 (84i:12071)


