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ABOUT THE COVER: THE CYCLOID AND JEAN BERNOULLI

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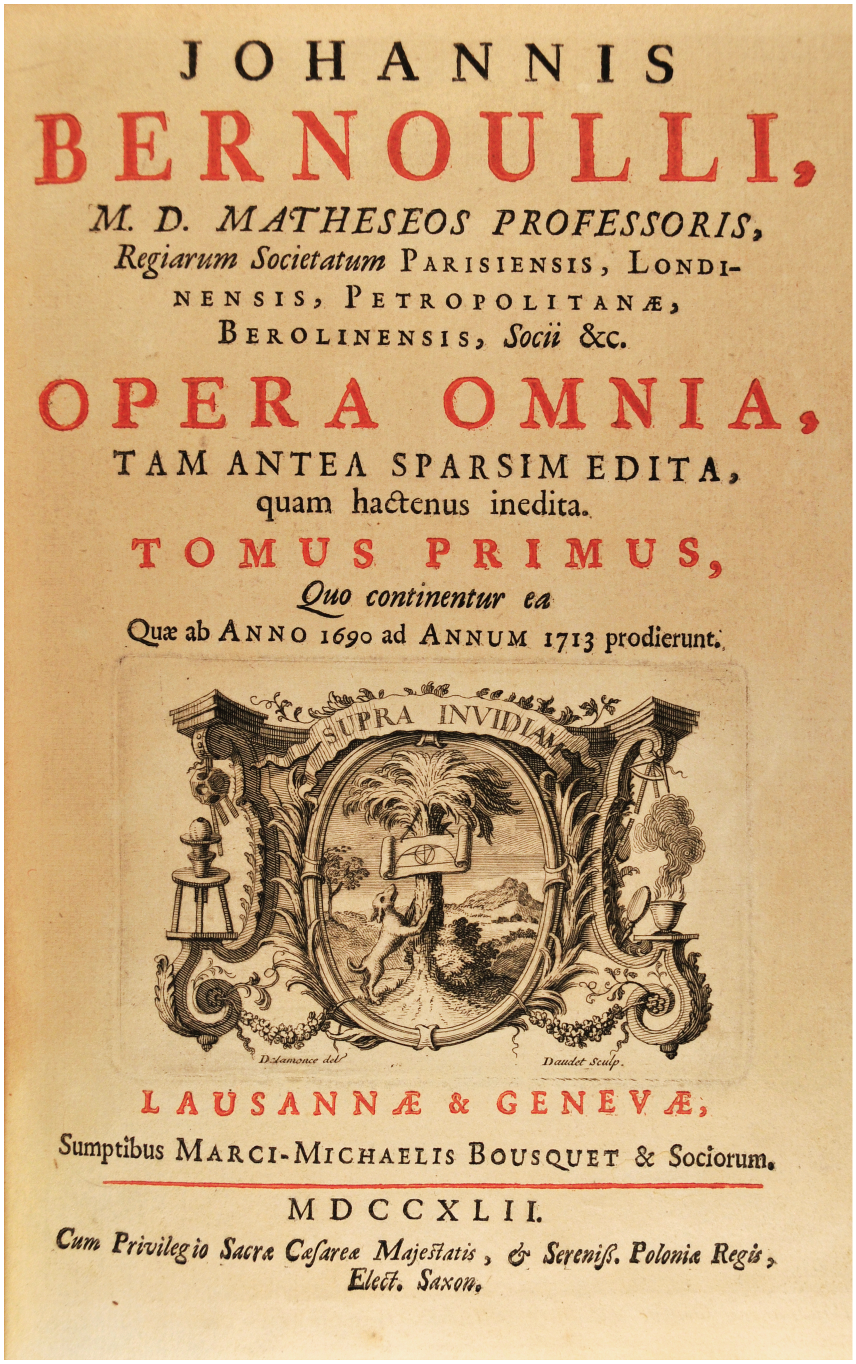
Johann (Jean) Bernoulli (1667–1748), the younger brother of Jacob (Jacques) Bernoulli, was a member of a large family of respected spice traders and scholars in Basel. Originally Flemish, they had fled to Switzerland to avoid religious persecution at the hands of the Spanish, who then occupied the Low Countries.

The 1742 *Opera Omnia* of Johann is a four-volume set that includes not only his mathematics but also work on fermentation and on the design of naval vessels. Handsomely produced by the Swiss firm of Bousquet, the first volume opens with a frontispiece portrait of Bernoulli (see Figure 1) followed by a colorful title page (see Figure 2). On that page there is a curious engraving of a dog with its front feet on the trunk of a tree—a palm, perhaps?—in a mountainous landscape, with the dog looking at a picture of a geometric figure on a large scroll attached to the tree. As it turns out, the figure is a cycloid. Similarly, on the facing page, there is the extravagantly elaborate oval engraving of the bust of Bernoulli. He is holding a rolled piece of paper showing the same figure, which appears as Figure 1 of Table XV, facing page 336 in volume 1. The illustration is used in finding the center of gravity of a sector of a cycloid and appeared in an article in the *Acta Eruditorum* of June of 1699, page 316. The figure also appears on page 609 of the award-winning article by Apostol and Mnatsakanian on cycloidal areas [1]. These illustrations suggest the large role this curve played in the work of Bernoulli. Just to add an element of extravagance, the dedication of the *Opera Omnia* is accompanied by a lavish engraving, if anything, outdoing the one of Bernoulli himself—but this time of the patron "Fridericus III Rex Borussiæ" (Prussia). Something seems wrong, however: Frederick III of Prussia was not king until the late 19th century, long after the publication of the Bernoulli work. It would have made sense were it

FIGURE 1. Frontispiece of Jean Bernoulli's *Opera Omnia*, 1742.



JOHANNIS
BERNOULLI,
M. D. MATHESEOS PROFESSORIS,
Regiarum Societatum PARISIENSIS, LONDINENSIS, PETROPOLITANÆ, BEROLINENSIS, *Socii* &c.
OPERA OMNIA,
TAM ANTEA SPARSIM EDITA, quam hactenus inedita.
TOMUS PRIMUS,
Quo continentur ea
Quæ ab ANNO 1690 ad ANNUM 1713 prodierunt.

LAUSANNÆ & GENEVÆ,
Sumptibus MARCI-MICHAELIS BOUSQUET & Sociorum.
MDCCXLII.
Cum Privilegio Sacræ Cæsareæ Majestatis, & Sereniss. Poloniæ Regis, Elect. Saxon.

FIGURE 2. Title page of Jean Bernoulli's *Opera Omnia*, 1742.

Frederick II (the Great) who became king in 1740, two years before the publication date of the *Opera Omnia*. It's a mystery. Perhaps engravings were interchangeable at that time: one picture fit all subjects. Much earlier this would not have been unusual. The famous Nuremberg Chronicle of 1493 was generously illustrated with pictures of cities throughout Europe. But sometimes the same picture was used for more than one city! It saved money.

The publisher, Marc-Michel Bousquet, appears to have been dazzled by titles (in spite of his being Swiss). There are acknowledgments of patronage not only by the King of Prussia but also by Emperor Charles VII of the Holy Roman Empire, and Friedrich August, King of Poland and Elector of Saxony.

Calculus students recognize the cycloid as the solution to the brachistochrone problem, the curve that allows a bead rolling down a trough in the form of the curve to reach the lowest point on the curve in the least amount of time. And, surprisingly, it is also the curve on which a bead rolling down from any point on the left side of the inverted arch of a cycloid reaches the bottom at exactly the same time as a bead rolling from any point on the opposite side—that is, it is also the solution to the tautochrone problem, the curve providing the "same time descent". In the other problem—the brachistochrone—the cycloid is the curve of "least time descent". These and similar problems were of widespread interest in the late 17th and early 18th centuries.

The cycloid is, of course, the curve traced out by a point on the boundary of a circlular disk as the disk is rolled, without slipping, along a straight line. Ordinarily, it would be shown as a series of arches, though in the two contexts above the curve has been reflected about the horizontal straight line. John Wallis thought the curve was known as early as 1451 to Nicholas de Cusa, but that claim has been questioned [6]. It was Galileo who gave it the name "cycloid" and investigated some of its properties around 1599. He was looking for curves of least time descent, though without much success. At the same time Mersenne, Roberval, and Torricelli became interested in the curve. Pascal made some real contributions to the subject, primarily in calculating the length of the curve and various volumes as it is rotated about axes (though he often used the French word for it, *roulette*). Torricelli correctly found the area under one arch to be three times the area of the generating circular disk. In a series of papers starting with "Problemata de Cycloide proposita mense Junii 1658" [5], along with correspondence between Huygens and A. Dettonville (a pseudonym used by Pascal—an anagram of Louis de Montalte, the name under which he published his *Lettres provinciales*), Pascal proved various results and published a challenge to others to replicate them. In 1658 Christopher Wren correctly calculated the arc length of one arch, $8a$, where a is the radius of the generating circle.

The first truly impressive achievements came along later. In 1673 the Dutch physicist-mathematician, Christiaan Huygens, published his book, *Horologium Oscillatorium*, a landmark in the history of science, in which he used the tautochrone (= isochrone) property of the cycloid. Huygens designed a pendulum clock that would keep good time by having the bob of the pendulum follow the path of a cycloid so that the clock became less dependent on being placed on a horizontal surface—a result particularly applicable to clocks on a ship subject to roll when out at sea. With the importance of the spice trade at that time, having a clock on a ship that could tell time accurately was a significant motivating force in applied mathematics. An interesting picture of Huygens's clock appears on page 413 of [3] and a photograph of such a clock (identified as a "Slingerklok naar Chr. Huygens") appears on a Dutch postage stamp.

Pascal's challenge to his colleagues was not the most memorable one involving the cycloid, however. That appeared in 1696 when Johann Bernoulli succeeded in showing that the cycloid is the solution of the brachistochrone problem, providing a faster descent between two points than Galileo's earlier straight line or broken line, or any other curve. Bernoulli sent a challenge (to prove that the cycloid is the solution) [2] to his older brother, Jacob, as well as to Leibniz, Newton, and l'Hôpital. Though Jacob Bernoulli was quick to respond to the challenge, his solution in finding the cycloid to be the curve of least time descent was rather clumsy and was not admired by his brother Johann whose solution displayed some striking insights. But in the end Jacob was the winner: his method widened the scope of the field to include isoperimetric problems and pointed in the direction of the development of the calculus of variations. (Relations between the two brothers—due to jealousy, perhaps?—were often strained.) Newton claimed that he did the brachistochrone problem in a few hours, but he is also reported to have grumbled that he did not like very much being teased by foreigners.

The years after Johann Bernoulli's important result were filled with further activity involving the cycloid. Echoing Huygens's result that the involute of a cycloid is again a cycloid but shifted along the horizontal line, it was shown that the evolute (locus of the centers of curvature) is also a congruent cycloid. The caustic of a cycloid is another cycloid (not congruent). And then there are the variants of the definition of the cycloid—the circle can roll around on the outside of the second circle (the epicycloid) or the inside (the hypocycloid). Investigations of all of these continued to interest Bernoulli and beautiful engravings of them appear in tables LIX to LXVIII in volume 3 of his *Opera Omnia*. There are even figures indicating that he was also curious about the generalizations achieved by rolling a regular polygon around the outside of another regular polygon, generating curves that have the general appearance of an epicycloid, but are not smooth and are indeed rather lumpy! It was clearly an age of cycloids. This fascination continued for many years, and there is a reference to the cycloid even in classic American literature: Herman Melville's *Moby Dick* (1851). Galileo had remarked that part of a cycloidal arch would make a nice-looking bridge. There are claims that the repeated vaults in the ceiling of Louis Kahn's Kimball Art Museum in Fort Worth, Texas, are cycloidal. Whatever they are, they are pleasing to look at. A large demonstration piece for the brachistochrone problem played a role in the science area of the Golden Gate International Exposition in San Francisco in 1940 [6]; science museums today have similar exhibits. And a small version of the model sits on the desk in my office.

Euler's name has not played a role in this account so far, but Johann Bernoulli was well known to have been Euler's teacher. So it should come as no surprise that Euler's first publication [4], written when he was eighteen, opens with: "It has been observed amongst geometers that the ordinary cycloid is an isochronous or tautochronous curve.... I marvel greatly that no one has yet considered hypotheses for the isochrones in media with other forms of resistance" (translation by I. Bruce). Euler then proceeded to solve that problem, but some of Euler's assumptions were incorrect, and that affected the final conclusion. So even the greatest of all 18th century mathematicians could make a mistake, at least when he was very young.

Euler went on to publish a masterpiece, his *Methodus inveniendi lineas curvas* (1744), which, with the later work of Lagrange, established a whole new field of mathematics, the calculus of variations.

The results on cycloids obtained prior to Bernoulli were for the most part done with Archimedean geometric methods, without the benefit of the calculus of Leibniz (1684) and Newton (1687). That all of the mathematicians mentioned, from Galileo up through Fermat, Descartes, Mersenne, Torricelli, and Roberval, found problems about the cycloid difficult is not surprising, but now these problems are standard exercises in calculus textbooks. It tells us something of the power of the calculus.

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