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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
DAVID NADLER

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Ngo, Bao Chau

Le lemme fondamental pour les algebres de Lie.

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The present paper, for which its author was awarded a Fields Medal, had achieved, even before publication, considerable fame and the Proceedings of the 2010 International Congress of Mathematicians in Hyderabad [World Sci. Publ., Hackensack, NJ, 2011] will contain two accounts of it, one by the author himself and one by James Arthur, a laudation delivered at the presentation of the prize. Both accounts are extremely instructive, and I refer the reader to them, as well as to two excellent accounts of the fundamental lemma available as preprints, one by T. C. Hales ["The work of Ngô Bao Châu", preprint, `arxiv.org/abs/1012.0382`] and one by D. Nadler ["The geometric nature of the fundamental lemma", preprint, `arxiv.org/abs/1009.1862`].

There is a great deal to be said about the fundamental lemma, about its origins, about the methods used to prove it and the developments that preceded the proof itself, and about its consequences or possible consequences, much more than could be accommodated in a normal review. No one is yet familiar with all this material. As a consequence, a good deal has been written about the lemma that, in my view, is misleading. I am convinced that anyone who wants to contribute to the central problems in the contemporary theory of automorphic representations, or, better, to *functoriality* and matters related to it, will need a better grasp of all these matters than any one person possesses at present. I shall try here to clarify this assertion, although this will entail a risk: not only of false prophecy but also of revealing my own ignorance. I understand the origins of the lemma and I believe I have as much insight into its possible consequences as anyone, but the proof itself, which exploits difficult tools and concepts from both modern algebraic geometry and topology, contains a very great deal of which I have only an uncertain understanding. The reader should take what I say about geometry or topology with a grain of salt.

The origins of the lemma are in the theory of Shimura varieties and in the theory of harmonic analysis on reductive groups over $\mathbb{R}$. This second source is analytic and algebraic, the theory of the spectral decomposition of invariant distributions on real reductive groups, a theory that we owe almost in its entirety to Harish-Chandra, although the basic idea, that the pertinent eigenfunctions are characters, was introduced in the context of finite groups by Dedekind and Frobenius. What was imposed on our attention by the theory of Shimura varieties and the trace formula was the understanding that for reductive algebraic groups there are two different notions of conjugation invariance: invariance and stable invariance. These are a result of two different kinds of conjugacy in, say $G(\mathbb{R})$, but more generally in $G(F)$, where F is a local field, archimedean or non-archimedean. One is conjugacy in $G(F)$ itself, the other is conjugacy in $G(\overline{F})$, where $\overline{F}$ is the (separable) algebraic closure of F. It was only as we began the study of the zeta-functions of Shimura varieties with the help of the trace formula that the importance of the distinction, its consequences, and the attendant difficulties were recognized. They led to the fundamental lemma.

The issue, at first, is less the fundamental lemma, which can take diverse forms, than its consequences, not only for Shimura varieties but more importantly for harmonic analysis, both local and global. With the fundamental lemma, it is possible to create a theory of endoscopy that reduces invariant harmonic analysis, even various forms of twisted-invariant harmonic analysis, on arbitrary reductive groups to stably invariant harmonic analysis on quasi-split groups. It is the latter in which the notion of functoriality is best expressed, and it is functoriality, still to a large extent conjectural, that is the source of the arithmetic power of representation theory and harmonic analysis. Specific forms of functoriality have already been used in the course of establishing Fermat's theorem and other conjectures of considerable interest to arithmeticians.

The fundamental lemma, once proved, offers two methods to attack functoriality: the first more immediate; the second much more encompassing. Although more limited, the first is of great importance, as it has offered to Arthur reasons for developing the general trace formula, which, thanks to him, has been given a chance to demonstrate the enormous power of nonabelian harmonic analysis, of which the trace formula is an expression, for arithmetic. The lemma allows global, and presumably also local, transfer of stable characters from the endoscopic groups H for a given group G provided with a twisting, perhaps trivial, to the group G itself. The best reference for this type of theorem will be Arthur's book *The endoscopic classification of representations: orthogonal and symplectic groups* [in preparation]. It promises to increase greatly the confidence of mathematicians at large in the notion of functoriality, even though the functoriality yielded directly by endoscopy is limited. I add that, in my view, the central issue in endoscopy is the theory with no twisting.

After the introduction of endoscopy, there were a good many years during which I did not pay much attention to the attempts to develop it, on one hand, by Waldspurger, Hales, and others, and, on the other hand, by Goresky, Kottwitz, and MacPherson. These contributions not only made possible the final proof of the lemma in the hands of Laumon and then Ngô, but also introduced ideas that will, I expect, play a major role in the continuing attack on functoriality.

The principal tools of Harish-Chandra in the development of harmonic analysis on real reductive groups and then, later, of Shelstad's treatment of endoscopy were the bi-invariant differential operators on the group. The spectral decomposition amounts to a spectral decomposition of this family of commuting operators on $L^2(G(\mathbb{R}))$. This is a local theory. Although a great deal of effort has been spent on non-archimedean fields, the theory has not reached the same stage, in good part because the spectral theory could not be reduced to one for a commutative family. My impression on studying the work of Waldspurger, Laumon and Ngô, without yet in any sense mastering it, is that the cohomology theory of perverse sheaves may offer a substitute, so that the possibilities offered by Waldspurger's reductions have by no means been exhausted.

Without any real knowledge of perverse sheaves as I began the study of Ngô's proof, and the earlier work with Laumon, and still only superficially informed, I am struck by the advantages of working with them. At the coarsest of levels, the orbital integrals provide over $\mathbb{R}$ or $\mathbb{C}$ the transfer that is dual to the transfer of characters from Cartan subgroups H of G, or better, although the theory has not been properly developed in this form even over $\mathbb{R}$, the transfer of characters implied by functoriality. Something similar will, I suppose, be true for non-archimedean fields, but it will be more delicate because some irreducible characters are not associated to a Cartan subgroup, for example, those associated to representations of the local Galois groups as tetrahedral representations. What, in my view, is taking place in Waldspurger's analysis, although I have yet to examine it with sufficient care, or even any care, is a reduction of the local analysis to the study of orbital integrals on Lie algebras, not over a local field but over a finite field, or, better expressed, in the context of algebraic geometry over a finite field. The asymptotic behavior described by the germs of Shalika becomes at this level a question of direct images of perverse images and their support, thus a behavior that is strictly geometric and strictly within the range of behavior encountered already in the study of these sheaves. I can imagine that the geometric information available through this translation might replace Harish-Chandra's study of the orbital integrals and their jumps to characters of G. Something similar to the jump conditions that Harish-Chandra met, and even something more subtle, might appear. I imagine that, when examining the possible behavior of the direct images with care, one will find behavior that can only be explained with the help of local Galois groups that admit surjective homomorphisms to relatively complex solvable groups. These matters will have to be studied on their own.

This kind of local information will be necessary if the program proposed for the utilisation of the stable trace formula—a formula available only after the fundamental lemma has been established—is to succeed in establishing functoriality. It is to be utilised in combination with the Poisson formula on the Steinberg-Hitchin base, an affine object introduced by myself with E. Frenkel and Ngô [Ann. Sci. Math. Québec **34** (2010), no. 2, 199–243]. The introduction of the Poisson formula was suggested by Ngô's use of the Hitchin base.

None of this explains the reasons for the success of Ngô nor for the earlier partial successes of Goresky-Kottwitz-MacPherson and Laumon-Ngô. Moreover, with the exception of Arthur's laudation, little attention has been paid in various expositions to the needs of specialists of the theory of automorphic representations, thus of those to whom the lemma itself is of the most interest and who may, like me, have little, if any, familiarity with stacks, perverse sheaves, or equivariant cohomology. So it may be worthwhile for me to have attempted to describe some glimpses of understanding that I have had while trying to penetrate their thoughts. I still have a long way to go and I am not certain that these glimpses are not will-o'-the-wisps. Waldspurger and one or two others may have clearer notions of the possibilities than I.

The fundamental lemma itself appears in the context of orbital integrals, thus integrals over the conjugacy classes $\{g^{-1}\gamma_G g\}$ defined by elements $\gamma = \gamma_G \in G(F)$, F a local field, for the present non-archimedean. For γ_G semisimple and regular, the conjugacy classes within the stable conjugacy class of γ_G are parametrized, in essence, by the elements of the abelian group $H^1(\mathrm{Gal}(\overline{F}/F), T)$, T the centralizer of γ_G. If κ is a character of this group, we may form $\sum \kappa(\gamma'_G)\mathcal{O}_G(\gamma'_G, f_G)$, where the sum over conjugacy classes is to be interpreted as a sum over $H^1(\mathrm{Gal}(\overline{F}/F), T)$, and f_G is the unity element of the Hecke algebra over G. Associated to κ is an endoscopic group, thus a quasi-split reductive group H, and to γ a stable conjugacy class $\{\gamma_H\}$ in H, for which we can form a stable sum $\mathcal{O}_H^{\mathrm{st}} = \sum \mathcal{O}_H(\gamma'_H, f_H)$, where f_H is the unity element in the Hecke algebra of H. The fundamental lemma, in its simplest and earliest formulation, is the equality of these two sums, up to a well-defined constant factor that will necessarily depend on the choice of Haar measure on G and H.

After Waldspurger's reduction, a new, but similar, equality appears with integrals over a set determined by an element γ, again often semisimple and regular, of the Lie algebra $\mathfrak{g}$ of G (or H) over F', again a local field but of positive characteristic, the ring of formal power series over a finite field k. Not having followed the developments over the years, I find the transition from one context to the other abrupt. My intuition is often brought up short. In addition, the proof of the fundamental lemma, like early proofs in local class field theory and occasionally elsewhere, is an argument from a global statement to a local statement, so that the function field F of a complete nonsingular curve X over k of which F' is a completion at some place v is introduced. G is replaced by a group over this new F and γ by an element of the Lie algebra $\mathfrak{g}$ of G, or more precisely by a section of the Lie algebra bundle defined by a G-bundle over X, a section that is allowed to have poles of large but finite order at a certain number, again large but finite, of points. It is in this context—especially difficult for those not sufficiently conversant with the notions of modern algebraic geometry—that the proof functions.

I was first disoriented by the appearance of Picard varieties in this context. They seemed to be of the usual type, thus closely related to abelian varieties. It was only after some time, when I noticed that the point of departure was the first cohomology group of a torus—thus a multiplicative group—the centralizer of γ, and that it was entirely possible that the transition from the local field F to the function field F of X and from Galois cohomology to étale cohomology or other cohomologies might entail the appearance of Picard varieties, that I began to feel more at ease. Galois cohomology groups have not been for me geometric objects. As descriptions of families of line bundles, thus of cohomology groups with values in GL(1) or, possibly, other abelian algebraic groups, Picard varieties (or stacks) may be representable—whether by varieties or by stacks—and thus subject to study by the usual methods of algebraic geometry. Once reoriented, I found it much easier to follow, at least superficially, the presentations by Ngô and others of the geometrical proofs of the fundamental lemma, in the final form as well as in the earlier forms.

Nevertheless, in Ngô's proof and in the reflexions of other authors that preceded it, there are several notions of which my grasp is tenuous: equivariant cohomology on the one hand and the apparently related notion of stacks on the other. Some aspects of the structure of the proof are quite clear. At a given place of X that is defined over k, in particular at the place with which we began, the orbital integrals, both for G and for H, can be interpreted as counts, although the count is a weighted count because centralizers of the elements γ interfere. One of the functions of stacks and equivariant cohomology, for those who understand them, is to take this weighting into account. That said, thanks to the passage to a global context, in the sense of algebraic geometry, thus to the passage to X and bundles over X, the counting, or rather the equality of two different counts asserted by the fundamental lemma, is replaced, in the spirit of the Weil conjectures and the Lefschetz formula, by an isomorphism of cohomology groups. The global count is, however, a sum over the points of X of local counts, so that, a global equality once established in general, it is necessary to return to X and to the section of the $\mathfrak{g}$-bundle that replaced the original γ, and to make choices that allow us to isolate the local contribution with which we began. Most of the effort is expended on the proof of the global cohomological statement, in the context of perverse sheaves for the étale cohomology and in the context of stacks.

I found it difficult to discover and keep firmly in mind the nature of the local count. There are at least two parameters at hand: the point of X and the point γ, which is now a section φ of the Lie algebra of a G-bundle E on X the total order of whose poles is controlled by a divisor D. The family $\mathcal{M}$ of these *Hitchin pairs*, (E, φ), is an essential element of the theory. The family of the classes in the Lie algebra of the group in question, G or one of its endoscopic groups H as the case may be, is the Hitchin base, a designation now familiar, thanks to Ngô, to a wide mathematical audience. The count is made over this base. Rather, the count is made, for both G and H, after a projection to this base. The domain of the projection is, to a first approximation, a scheme whose points are, first, a G-bundle on the given base X and, second, the section γ. So, implicit in the discussion is, I suppose, the existence of moduli spaces or stacks and an understanding of the cohomology of perverse sheaves defined on them. Most of this, and much else, I have to take on faith at present.

The Hitchin base is, as an algebraic variety over k, an affine space. The count on the fiber is made indirectly, through the direct images of the cohomology of the fiber. This fiber has, I believe, two important features. One feature it shares with the usual Picard varieties, namely an action of a very large connected group, sometimes an abelian variety; this large group is defined over the Hitchin base. If I understand correctly the explanation in Ngô's Hyderabad lecture, an important consequence is that the action of the full group, a Picard group (rather stack!) $\mathcal{P}$ in the sense of Ngô, on the cohomology of the fibers is defined through a discrete quotient, denoted $\pi_0(\mathcal{P})$ by Ngô, a possibility that is certainly plausible from a topological point of view. This discrete quotient is closely related to the Galois cohomology groups $H^1(\mathrm{Gal}(\overline{F}/F), T)$ with which we began. These things are well explained in Ngô's Hyderabad lecture, where it is also explained that the local discrete quotients can be patched together, but in the étale topology, to form a sheaf of abelian groups. It is somewhat comforting, and perhaps not altogether incorrect, if we think of this as a patching in the étale topology of the various $H^1(\mathrm{Gal}(\overline{F}/F), T)$, defined for widely varying tori T. In any case this allows the discrete quotient and its characters to be introduced globally, something that was done in a different manner in the original formulation of the lemma.

The result is a sheaf over the Hitchin base that permits an action of the group $\mathcal{P}$. Since $\mathcal{P}$ acts on the fibres over the base, its action defines an action on the direct image of the cohomology on the Hitchin base, an action that factors through $\pi_0(\mathcal{P})$. Consequently the direct image can be decomposed as a direct sum with respect to the characters κ of $\pi_0(\mathcal{P})$. The principal theorem of Ngô, at least in connection with the fundamental lemma, is to establish that each component of the direct sum is isomorphic to a similar component for an endoscopic group H over X, a group defined by the character κ.

There is a fluidity in the development of the proof that Ngô captures in his various expositions. Ideas appear and reveal themselves as suggestive but ultimately inadequate, and then reappear in a different, often more difficult, guise. It is probably impossible to understand the final proof without some feeling for these initial stages, for equivariant cohomology in all its guises, and, above all, for the geometry of the Hitchin fibration. I certainly have a long way to go, but I find the relatively concrete form in which this fibration is used by Laumon-Ngô in the proof of a special case of the fundamental lemma a helpful guide to the general case.

Since the Hitchin fibration and its properties are basic, a word or two about its construction may not be inappropriate. For a vector bundle, thus for a GL(n)-bundle, one can associate to the section γ, or better to the point a in the Hitchin base, a matrix-valued function on X, and to each point $x \in X$, the n points in an n-dimensional space given by its eigenvalues. As x varies, these points trace out a curve, an n-fold covering $Y_a = Y_\gamma$ of X. With γ we can introduce, at least in favorable circumstances, more: for each point x and each of the eigenvalues, a line, the eigenspace corresponding to the eigenvalue. Thus the section γ defines a line bundle on Y_a. There are questions that arise at the points where the eigenvalues are multiple, but we do see line bundles on the horizon and therefore, perhaps, abelian varieties and cohomology groups in degree 1, groups related to those with which endoscopy began. The abelian varieties are a sign that, in the new context, these cohomology classes appear as line bundles that give rise to representable functors, whose points can be described geometrically. The Hitchin fibration, as defined by Ngô, provides similar constructions for a general group. Even in the original form, the eigenvalues associated to γ define at each point of X a diagonal matrix, but as the order of the eigenvalues is not prescribed, it is in fact only the conjugacy class of this diagonal matrix that is determined.

At the level of groups we cannot, so far as I know, ordinarily find a map from conjugacy classes to matrices that is inverse to that from matrices to conjugacy classes, but at the level of Lie algebras, low characteristics aside, we can. For example, for the group SL(2), the conjugacy class is given, at least at the regular elements by the determinant, a, and the representative matrix for this class can be taken to have diagonal elements 0 and off-diagonal elements 1 and a. There are, I believe, various such lifts. Ngô uses the one associated to the name of Kostant. Our original description of the spectral curve Y_a was deliberately vague about its form at those points where eigenvalues coincide and it is best here to pass over in silence the difficulties they entail in Ngô's definitions. They entail technical difficulties that I have not yet made any attempt to understand. Indeed, I am not much beyond the introduction to his paper. In any case, what results is a lift not only of the regular conjugacy classes of the Lie algebra to the Lie algebra itself, but an abelian group over these lifts. It is closely related to the centralizer of the lifts and yields a fibration in groups over the Hitchin base. The dimension of the fibers is the rank of G. In the definition of the Picard variety (stack) relevant to the Hitchin fibration and to Ngô's analysis, the bundles associated to this fibration in groups replace the line bundles of the classical theory. I have to remind myself constantly that there are two parameters at play in this fibration: the base a, given by the class of γ, and a point x of X, at which γ is essentially an element in the Lie algebra of G, say over the residue field or over the coordinate ring at x.

As already observed, the argument for the proof of the fundamental lemma proceeds in two stages: first, for a fixed a and all of X, but fortunately only for well-chosen a; second, for a suitable X and a suitable point x of X. We have already described the projection at the first stage, from the total space of the Hitchin fibration to the Hitchin base, and the decomposition of the direct image according to the characters κ of the Picard stack.

There is an equality of sheaves over X to be proven at the first stage. There are two issues in the proof of the equality: the support of the relevant direct images, and the equality on this support. An endoscopy group is so defined that there is a morphism of the Hitchin base $\mathcal{A}_H$ to $\mathcal{A}_G$. So we can compare the direct image of a sheaf on $\mathcal{A}_H$ with a sheaf on $\mathcal{A}_G$. The sheaf on $\mathcal{A}_G$ is defined by the part of the direct image of the sheaf associated to the character κ. For H, one does the same thing, but the character for H is taken to be trivial. If H is associated to κ, it has first to be shown that the direct image of the κ-component for G is supported on the image of the Hitchin base for H. This is, in principle, a consequence of the definitions, but it is not an easy consequence. Indeed, the final proof is tremendously daunting.

Those of us with less than adequate facility with the concepts can best begin with the theorem for unitary groups proved by Laumon-Ngô, because in [G. Laumon and Ngô Báo Châu, Ann. of Math. (2) **168** (2008), no. 2, 477–573; MR2434884 (2009i:22022)] not only does the Picard stack appear in its primitive form in terms of the spectral curve Y_a, but, in addition, the proof of the necessary *homotopy lemma*, which is used to deal with the problem of support, appears to be at an altogether different level of difficulty than the *support theorem* of the paper under review. In [G. Laumon and Ngô Báo Châu, op. cit.], both G and H are unitary groups. Since a unitary group is a form of $\mathrm{GL}(n)$, the concept of spectral curve has a more immediate geometric content and there is a more direct relation between the Hitchin fibrations of G and H that appears to simplify the arguments considerably.

I have already adumbrated the final step of the proof. If the curve X and an element a of the Hitchin base are given, they define locally at any point x of X the elements for the original statement of the fundamental lemma for the Lie algebra, an element of the local Lie algebra and a group $G(\mathcal{O}_x)$. Moreover, the equality of a κ-component of the direct image at a with a direct image for an endoscopic group H can be interpreted, thanks to the Grothendieck-Lefschetz theorem, as an equality of the product over the points of X of two counts, one for κ-components on G and one for H. If we can choose x, X, and a so that they reproduce any arbitrarily given local data and if X and a are also chosen such that the fundamental lemma is true at all points $x' \neq x$ of X, we can cancel all terms in the product but those at x and deduce the desired equality at that point. We cannot, apparently, expect to choose X such that the fundamental lemma is utterly obvious away from X, but it can be so chosen that it is accessible to direct computation. To establish the existence of X and a with the necessary properties has required, both in the present paper and in the earlier paper on unitary groups, very sophisticated algebro-geometrical methods. It is also important for its existence that the poles of the section defining a are allowed to grow in number.

For the unitary groups, the very last step, the deduction of the fundamental lemma outside of x from the properties of X and a appears almost an elementary exercise in geometry over finite fields. This is not so in general. Further struggles with perverse sheaves await the reader.

It is clear that, for the majority of specialists in nonabelian harmonic analysis and representation theory, thus, in particular, for specialists in the theory of automorphic representations and the associated arithmetic, certainly for me, it will take more than a few weeks, or even a few months, to assimilate the techniques from contemporary algebraic geometry that are required for the proof of the fundamental lemma. How long it might take geometers to understand fully the questions posed by the arithmetic and the analysis, I hesitate to guess. This might be easier.

Representation theory has a briefer and, in some respects, narrower history, but it is less familiar to the majority of mathematicians. Time will tell.

Since, as I intimated at the beginning of this review, the fundamental lemma is an essential and fundamental contribution to a theory that will not be developed by specialists in algebraic geometry alone, there will be a need for further, more accessible expositions of the methods of this paper and those that preceded it, with examples, even very simple examples, and with considerably more explanation of the geometric intuition implicit in the abstract theory. An index to definitions and symbols would also be welcome! The present paper is 168 pages long and these pages are large and very full. An exposition genuinely accessible not alone to someone of my generation, but to mathematicians of all ages eager to contribute to the arithmetic theory of automorphic representations, would be, perhaps, four times as long, thus close to 700 pages. It would, I believe, be worth the effort.

R. P. Langlands

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Laumon, Gerard; Ngo, Bao Chau

Le lemme fondamental pour les groupes unitaires.

Annals of Mathematics. Second Series **168** (2008), *no.* 2, 477–573.

Let G be an unramified reductive group over a non-Archimedean local field F. The so-called Langlands Fundamental Lemma (see $(*)$ below when G is a quasi-split unitary group) is a family of conjectural identities between orbital integrals for $G(F)$ (the group of F-points of G) and orbital integrals for endoscopic groups of G. In the article, the authors prove the Langlands Fundamental Lemma in the case when F is a finite extension of $\mathbb{F}_p((t))$ and G is a unitary group of rank strictly less than p. J.-L. Waldspurger showed in [J. Inst. Math. Jussieu **5** (2006), no. 3, 423–525; MR2241929 (2007h:22007)] that this case implies the Langlands Fundamental Lemma for unitary groups of rank strictly less than p when F is any finite extension of $\mathbb{Q}_p$. Note that for global applications the condition "p large with respect to the rank" is satisfied at almost all finite places.

Let $G = \mathrm{U}(n)$ be the quasi-split unitary group over F of rank n. Let $n = n_1 + n_2$ be a nontrivial partition of n, and let $H := \mathrm{U}(n_1) \times \mathrm{U}(n_2)$ be the endoscopic group of G which corresponds to (n_1, n_2). Let $\delta = (\delta_1, \delta_2)$ be an elliptic regular semisimple element of $H(F)$, and let $T = T_1 \times T_2 \subset H$ denote the centralizer of δ in H. For $i = 1, 2$, the group T_i is an F-anisotropic maximal torus in $\mathrm{U}(n_i)$. One can embed T as a maximal torus in G. Then let γ denote the image of δ by the embedding. The element γ is elliptic semisimple. Assume that it is regular in G.

Let F' be the quadratic unramified extension of F. The set of conjugacy classes in the stable conjugacy class of γ in $G(F)$ is in natural bijection $\lambda \leftrightarrow \gamma^\lambda$ with the finite group

$$\Lambda = \Lambda_r := \{\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{Z}/2\mathbb{Z})^r :\ \lambda_1 + \cdots + \lambda_r = 0\},$$

where r is the rank of the maximally split F'-torus contained in the centralizer of γ in $\mathrm{GL}(n, F')$. Similarly, the set of conjugacy classes in the stable conjugacy class of δ in $H(F)$ is in natural bijection $\lambda \leftrightarrow \delta^\lambda = (\delta_1^\lambda, \delta_2^\lambda)$ with the subgroup $\Lambda^H := \Lambda_{r_1} \times \Lambda_{r_2}$ of Λ, where for $i = 1, 2$ the integer r_i is the rank of the maximally split F'-torus contained in the centralizer of δ_i in $\mathrm{GL}(n_i, F')$. (One has $r_1 + r_2 = r$.) For each $\lambda \in \Lambda$, the centralizer T^λ of γ^λ is an inner form of T, and hence is isomorphic to

T. Similarly, for each $\lambda \in \Lambda^H$, the centralizer S^λ of δ^λ is an inner form of T, and hence is isomorphic to T.

Let $\mathcal{O}_{F'}$ denote the ring of integers of F'. Let $K = K_n := G(F) \cap \mathrm{GL}(n, \mathcal{O}_{F'})$ and $K^H := K_{n_1} \times K_{n_2}$ denote the standard maximal compact subgroups of $G(F)$ and $H(F)$, respectively. Let 1_K and 1_{K^H} denote the characteristic functions of K and K^H, respectively. Let $O_{\gamma^\lambda}(1_K)$ (with $\lambda \in \Lambda$) and $O^H_{\delta^\lambda}(1_{K^H})$ (with $\lambda \in \Lambda^H$) be the orbital integrals defined by

$$O_{\gamma^\lambda}(1_K) := \int_{T^\lambda(F)\backslash G(F)} 1_K(g^{-1}\gamma^\lambda g)\frac{g}{t^\lambda},$$

$$O^H_{\delta^\lambda}(1_{K^H}) := \int_{S^\lambda(F)\backslash H(F)} 1_{K^H}(h^{-1}\delta^\lambda h)\frac{h}{s^\lambda}.$$

(The Haar measures g and h on $G(F)$ and $H(F)$ are normalized so that the groups K and K^H have volumes 1. A Haar measure on $T(F)$ is fixed and then transported in t^λ, s^λ via the above-described isomorphisms between T and T^λ and between T and S^λ, respectively.) Let $\kappa\colon \Lambda \to \{\pm 1\}$ be a character which has kernel equal to Λ^H. The Langlands-Shelstad κ-orbital integral is defined to be the following linear combination of orbital integrals:

$$O^\kappa_\gamma(1_K) := \sum_{\lambda\in\Lambda} \kappa(\lambda) O_{\gamma^\lambda}(1_K),$$

while the Langlands-Shelstad stable orbital integral is

$$SO^H_\delta(1_{K^H}) := \sum_{\lambda\in\Lambda^H} O^H_{\delta^\lambda}(1_{K^H}).$$

Then the Fundamental Lemma is the identity (conjectured by Langlands and Shelstad):

$$(*) \qquad O^\kappa_\gamma(1_K) = \Delta(\gamma,\delta) SO^H_\delta(1_{K^H}),$$

where the transfer factor $\Delta(\gamma,\delta)$ equals, up to a sign, $|D_{G/H}(\gamma)|^{1/2}$ at the power of the order of the residue field of F.

The proof uses as a key ingredient a geometric interpretation of the elliptic endoscopy theory in terms of the Hitchin fibration. This interpretation due to the second author is valid in great generality [see Invent. Math. **164** (2006), no. 2, 399–453; MR2218781 (2007k:14018)]. It was also used later in his proof of the Langlands-Shelstad Fundamental Lemma for arbitrary Lie algebras. More recently, P.-H. Chaudouard and the first author extended the latter and obtained a general proof of the Arthur Weighted Fundamental Lemma.

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Dat, Jean-Francois

Lemme fondamental et endoscopie, une approche geometrique (d'apres Gerard Laumon et Ngo Bao Chau.

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The "fundamental lemma" of Langlands and Shelstad is a family of combinatorial conjectures which relates orbital integrals of different (but related) algebraic groups over a local field, and which plays a fundamental role in the (global) Langlands program, specifically, the functoriality principle and the expression of L-functions of Shimura varieties in terms of automorphic L-functions. In this paper, the author gives an introduction to the fundamental lemma, and in particular to recent results of Laumon and Ngô [G. Laumon and Ngô Báo Châu, "Le lemme fondamental pour les groupes unitaires", preprint, `arxiv.org/abs/math/0404454`, Ann. of Math. (2), to appear; Ngô Báo Châu, Invent. Math. **164** (2006), no. 2, 399–453; MR2218781 (2007k:14018)], who proved the fundamental lemma for unitary groups. This survey is a useful companion to the articles of Laumon and Ngô.

It consists of three parts. The first one is an introduction to endoscopy: The author recalls the context of the Arthur-Selberg trace formula, and why one wants to "stabilize" it, i.e. to replace the sum over conjugacy classes by a sum over stable conjugacy classes (if the derived group of the group G in question is simply connected, then elements over the base field are stably conjugate if they are conjugate over the algebraic closure). Furthermore, the pre-stabilization due to Langlands and Kottwitz; endoscopy and global transfer; and local transfer and the fundamental lemma are discussed. Results of Waldspurger and others show that to prove the fundamental lemma (at least for unitary groups) it is enough to prove the Lie algebra version in the function field case with large residue characteristic. Finally, in the case of unitary groups the statement of the fundamental lemma is made entirely explicit. Both sides of the desired equality can be understood as numbers of certain lattices in some vector space.

The second part deals with the Hitchin fibration and the trace formula for general reductive groups. After some remarks about algebraic stacks, the Hitchin space and the Hitchin fibration are introduced. Here the author works with an arbitrary connected reductive group, but always discusses the case of a unitary group as an example. The underlying idea of the proof of Laumon and Ngô is that the number of points of the Hitchin moduli spaces formally looks like the geometric side of the trace formula. On the other hand, one can try to rewrite the number of points using the Grothendieck-Lefschetz formula. That is why it is useful to analyze the cohomology of the Hitchin fibration. The second part concludes with the statement of a conjecture (for all classical groups) of Laumon and Ngô which expresses the sought-for cohomological identity (in terms of certain graded perverse sheaves).

In the third part, the theory explained in the second part is analyzed in the special case of unitary groups. Here, all the notions can be made quite explicit. The author then comes back to the conjecture stated at the end of the second part and explains its proof by Laumon and Ngô. Its basic idea is that the Hitchin fibration provides a natural deformation of the complicated fibers to ones with less severe singularities which can be used to reduce the proof of the fundamental lemma to a very simple situation where it can be checked directly more easily.

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Laumon, Gerard

Aspects geometriques du Lemme Fondamental de Langlands-Shelstad.

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This paper is a survey on recent results on the so-called fundamental lemma. The fundamental lemma is a collection of conjectures of a combinatorial nature which relate orbital integrals over p-adic reductive groups. These conjectures are a cornerstone in the Langlands program, because they are needed in order to "stabilize" the Arthur-Selberg trace formula, i.e.—roughly—to replace conjugacy classes by stable conjugacy classes, which—again roughly—means considering conjugacy over the algebraic closure of the ground field. A stable trace formula is useful for instance to establish instances of Langlands functoriality, or to compute the Hasse-Weil zeta function of Shimura varieties.

The author briefly explains the notions coming into the formulation of the fundamental lemma: (stable) orbital integrals, κ-orbital integrals, the Langlands dual group, and endoscopic groups, and states the fundamental lemma as formulated by Langlands and Shelstad. All notions are made very explicit in the case of a unitary group, where the two sides of the conjectural equation can basically be expressed as numbers of lattices satisfying certain conditions.

The second half of the note is dedicated to recent results, which heavily use methods from algebraic geometry; the quantities appearing in the fundamental lemma can be related to the numbers of points of affine Springer fibers, and hence, by the Grothendieck-Lefschetz fixed point formula, to their cohomology. M. Goresky, R. E. Kottwitz and R. D. MacPherson studied this cohomology, and conjectured that it is pure [see Duke Math. J. **121** (2004), no. 3, 509–561; MR2040285 (2005a:14068)]. They were able to prove purity under additional (quite restrictive) hypotheses, and derived instances of the fundamental lemma from this. Their theorem applies to every group in question, but for no group does one get the complete fundamental lemma.

On the other hand, Laumon and Ngô were able to prove the complete fundamental lemma for unitary groups. A sketch of their approach is given in the final section. The basic idea is to consider deformations of affine Springer fibers with less severe singularities. More precisely, they first replace affine Springer fibers with compactified Jacobians, and then consider deformations of the latter given by the Hitchin fibration. See also Ngô Báo Châu's paper [in *International Congress of Mathematicians. Vol. II*, 1213–1225, Eur. Math. Soc., Zürich, 2006; MR2008f:14023].

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