BOOK REVIEWS

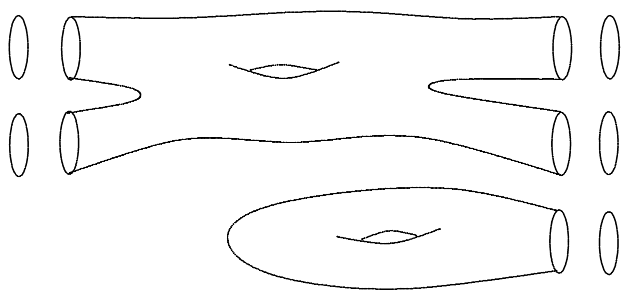
BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, April 2012, Pages 337–345
S 0273-0979(2011)01351-9
Article electronically published on September 13, 2011

Homotopy quantum field theory, by Vladimir Turaev, EMS Tracts in Mathematics, 10, European Mathematical Society (EMS), Zurich, 2010, xiv+276 pp., hardcover, \$78.00, ISBN 978-3-03719-086-9

Homotopy quantum field theory (HQFT) is an offshoot of topological quantum field theory (TQFT) and uses many of the methods and intuitions of that older theory. It seems therefore useful to start this review by recalling what that theory looks like.

The origins of TQFT were in theoretical physics, partially as simple models for some of the more structured field theories. The idea was to model the evolution of a *state space* of a manifold as the manifold changed along a cobordism. This was axiomatised by Atiyah, and then gradually a fairly definitive form of the axiom system was settled on. This needs a bit of categorical language, namely that of monoidal categories, so let us start by looking at the prime example of relevance to TQFTs, namely the category of d-manifolds and cobordisms between them.

This category has d-dimensional oriented smooth closed manifolds as its objects and between two such, X_0 and X_1, a morphism is, roughly, a $(d+1)$-dimensional oriented smooth manifold, M, whose boundary is the disjoint union of X_0 and X_1. This M is a *cobordism* from X_0 to X_1, and we picture it something like this:



$X_0 = S^1 \sqcup S^1 \qquad\qquad M : X_0 \longrightarrow X_1 \qquad\qquad X_1 = S^1 \sqcup S^1 \sqcup S^1,$

where the picture corresponds to a case where $d = 1$, so the cobordism is a surface with boundary $X_0 \sqcup X_1$ and where orientations match in a sensible way. This data is to make a category, so we need an associative composition operation—which

2010 *Mathematics Subject Classification.* Primary 57-02.

comes from gluing cobordisms end to end—plus identities. For the identities we have to invoke the term "roughly" in the above. We should have said something like "diffeomorphism classes of cobordisms relative to their boundaries" or similar. Of course, if $M : X_0 \to X_1$, then there is an opposite cobordism $-M$, going from X_1 to X_0 constructed by changing the orientation of M. Here is not the place to go into details of cobordisms, so if the reader wants more, a detailed treatment is in various places in the literature, for instance, in Kock's book, [4]. (We will usually abuse terminology and simply say the morphisms are cobordisms.) Another remark is that in some sources, the convention is used of speaking about the manifolds as being $(d-1)$-dimensional, so the cobordisms then are d-dimensional.

This category $(d+1)$- Cob, has a monoidal structure given by the coproduct of manifolds and similarly, of cobordisms (that is, disjoint union of the manifolds), and between them disjoint unions of the corresponding cobordisms. A monoidal category is one with a monoid-like multiplication on both objects and morphisms. The main classical example is that of the category, Vect, of (finite-dimensional) vector spaces over a field $\Bbbk$ together with linear transformations between them as morphisms and tensor product as the monoid-like multiplication. Between monoidal categories, it is usual and natural to consider monoidal functors, which, not surprisingly, are functors which respect the multiplications. (There are various strengths of these, but that will not concern us for the moment, as we just need the idea.) As Vect is "well known", "well behaved", and has a load of useful structure for extracting invariants of its objects and morphisms, it is not uncommon to compare a given monoidal category, $(\mathcal{C}, \otimes)$, with it, that is, to look for, and to study, representations of $\mathcal{C}$ in Vect. These are just monoidal functors. In particular, the categorical form of a TQFT is just the representation of some $(d+1)$- Cob in Vect,

$$Z : (d+1)\text{- Cob} \to \text{Vect},$$

or in any other similar nice monoidal category, such as a category of (projective) modules over some interesting ring.

Of course, it is easy to specify the structure encoded in this slick form in more elementary terms, so for each d-manifold X (and I will not repeat "smooth", "oriented", etc.), we will have a finite-dimensional vector space $Z(X)$, and as Z respects the monoidal structure, $Z(X \sqcup Y) \cong Z(X) \otimes Z(Y)$. To each cobordism, $M : X_0 \to X_1$, we will have $Z(M) : Z(X_0) \to Z(X_1)$, a linear transformation, and to the empty manifold, $\emptyset$, we will assign the ground field $\Bbbk$ considered as a 1-dimensional vector space. (This last bit of structure is needed as $\emptyset$ is the monoidal unit in $(d+1)$- Cob, $\Bbbk$ is that in Vect, and Z must preserve units.)

How are TQFTs used? The usual first example of the use of this type of structure is to consider a closed $(d+1)$-manifold, M as a cobordism, $M : \emptyset \to \emptyset$, so, applying Z will give $Z(M) : \Bbbk \to \Bbbk$, which, of course, being a linear map, corresponds to multiplication by an element of $\Bbbk$. That element will be an invariant of M, so a TQFT gives a numerical invariant on closed manifolds. (When we get back to discussing Turaev's ideas and the theory in his book, we will see other deeper applications as well.)

Another point to note before we look at HQFTs is that, for low values of d, one can classify TQFTs up to isomorphism. For instance, for $d = 1$ a TQFT, Z, corresponds to a commutative Frobenius algebra, which, in fact, is just $Z(S^1)$, the vector space corresponding to the circle, and in which the algebraic structure mirrors the geometric pictures of cobordisms. There is a related categorical description for $d = 2$. There are also various well-known constructions of TQFTs from groups and algebras, and via representation theory. In fact, the links with techniques and ideas from representation theory recur throughout the subject.

The idea of homotopy quantum field theory is to "decorate" the manifolds and cobordisms with some extra structure. This extra structure is in the form of a *characteristic map* to some fixed topological space. (In his book, Turaev denotes this space by X, but for the purposes of this review, we will use B. The letter B recalls (i) that the space acts somewhat as a "background" structure, or a "base" for the theory, and (ii) for most of the examples this fixed space will be the classifying space of a (discrete) group, and these are usually denoted BG.) This space, B, is to be a connected CW-space with a base point $*$. The objects of study in the HQFT are then maps $g : X \to B$, where X is a d-dimensional manifold (usually assumed to be smooth, although everything seems to work for topological or PL-manifolds as well). Further, X is to be closed, oriented, and pointed, that is, every connected component of X is provided with a base point, and the characteristic map, g, is to send these base points to $*$. The pair (X, g) is then called a *B-manifold* or, if more precision is needed, a d-dimensional B-manifold. This leads naturally to:

- A *B-homeomorphism* of B-manifolds, $(X, g) \to (X', g')$, is an orientation preserving diffeomorphism, $f : X \to X'$, such that $g'f = g$, and f preserves base points.
- A *B-cobordism*, (M, X_0, X_1, g), of B-manifolds, (X_0, g_0) and (X_1, g_1), consists of a $(d + 1)$-cobordism, M from X_0 to X_1, together with a map, $g : M \to B$ sending the base points of X_0 and X_1 to $*$ in B. The restriction of g to X_0 and X_1 gives the characteristic maps of the B-manifolds, (X_0, g_0) and (X_1, g_1), respectively.

Clearly, given any two d-dimensional B-manifolds, we can form their disjoint union, and this suggests that there should be a symmetric monoidal category that would have d-dimensional B-manifolds as its objects, some sort of morphisms constructed from $(d+1)$ B-cobordisms between them, and then disjoint union as the monoidal structure, and the idea would then be to study this monoidal category via its monoidal representations in some nice category such as Vect. That is a possible route, and the resulting monoidal representations are what are called the *homotopy quantum field theories* of the title of this book, but, in fact, that is not the route taken by the book, so we will not follow up that idea just now.

Before we start outlining the contents of the book, let us pause, ask some questions, and review some history.

First you may ask, "Why should this extension of TQFTs to HQFTs be a 'good thing'?" Also, "I am happy enough with thinking of TQFTs, but how should I interpret the objects being studied by HQFTs? They seem more artificial than the manifolds I am used to." The idea of finding invariants of manifolds from TQFTs has been quite successful. That is not to imply that there are loads of incredible new invariants that have been dreamt up from TQFTs—rather that TQFTs have made sense of the various invariants that were previously known. (The analogy may be a bit like the classical invariants, that were the Betti numbers, and which made a lot more sense and became more calculable once the idea of (co)homology groups was around.) To construct TQFTs, and thus some invariants for closed manifolds, one starts with some algebra or group and uses either a labelled triangulation or its dual or cohomological methods to build for each manifold a vector space. In the case that the input used for the labels is a (discrete) group, G, the vector space associated to X is typically related to isomorphism classes of principal G-bundles on that manifold. With an HQFT, taking B to be an Eilenberg–Mac Lane space, $K(G,1)$ (so a classifying space of G, with $\pi_1(B) \cong G$, and all other homotopy groups trivial), each (X,g) gives a principal G-bundle on X, so using this approach you are retaining more structure with which to "play", from the start. (This can be viewed as a form of partial *categorification*, where a set of objects is being replaced by a groupoid whose set of connected components is the previously considered set, but this is not being fully exploited in the present set-up. There is sufficient structure around already to give a rich crop of results, without that extra layer being explicitly exploited. This reviewer feels that some of the results proved here, however, may have simpler proofs if the process of categorification is exploited more fully.)

Topological quantum field theories correspond to B-HQFTs in which B is simply a single point, so we can hope that, in the HQFT setting, the structure will be richer and clarify some aspects of the former theory. Changing the "background", B along a continuous map $B \to B'$ will induce functorial transformations of the theories and will allow more flexibility in the methods of comparing or calculating invariants. We thus have more tools, and a natural interpretation of invariants, at least in certain cases.

Finally, before we start to describe the structure and contents of the book, we should say something of the history of the subject. Turaev made available two preprints ([10] in 1999 and [11] the following year) in which he described a version of the theory in some detail. These two preprints form the basis for some of the book, but have been extensively reworked and extended. At about the same time, Brightwell and Turner in [1] considered the representations of the *homotopy surface category* of a simply connected space. This was effectively a version of an HQFT for $d = 1$ and for a base which was simply connected, so that one could imagine taking $B = K(G,2)$, an Eilenberg–Mac Lane space (so $\pi_i(B) = 0$ if $i \neq 2$, whilst for $i = 2$ we have a group, G, which by necessity must be abelian). Shortly afterwards in 2001, Rodrigues [8] showed how to use a monoidal category approach to define HQFTs, but in the process this suggested that one of Turaev's original axioms from [10] was not optimal, and Rodrigues proposed an amended version, which is the one presented in this monograph. He also showed that replacing an arbitrary B by a space of the same homotopy $(d+1)$-type, such as the corresponding Postnikov section of B, resulted in equivalent theories. (Some other independent approaches to parts of the algebraic side of the theory will be mentioned towards the end of the review.)

Turning now to the contents, the main aim of this monograph is to show the existence of $(d+1)$-dimensional HQFTs with target a $K(G,1)$ for $d = 1$ and 2. Just as TQFTs in these dimensions can be analysed both algebraically and categorically, so analogous descriptions can be given for the resulting HQFTs yielding interesting variants of the types of algebra encountered in the simpler situations.

In most of the book, the restriction to B being a $K(G,1)$ applies, but initially the theory is developed for a general B. After a brief introduction to the terminology of cobordisms, etc., the axioms for a $(d+1)$-dimensional homotopy quantum field theory τ are given. These take the form of an assignment of structure to B-manifolds, B-homeomorphisms, and B-cobordisms in a way that is very closely modelled on the analogous set-up for a TQFT. Again, the axioms look like those of a TQFT, except there is an "additional" homotopy axiom, namely that if $(M, g : M \to B)$ is a $(d+1)$-B-cobordism, the homomorphism, $\tau(M, g) : \tau(X_0, g_0) \to \tau(X_1, g_1)$, in the notation we use earlier, only depends on the *homotopy class* of g relative to the boundary ∂M of M.

The first chapter continues by looking at some generalities on HQFTs, the category of HQFTs for a given dimension and background B, and then operations such as direct sum and tensor product on them. This chapter continues by exploring the construction of a $(d+1)$-dimensional B-HQFT from a given cohomology class, $\theta \in H^{d+1}(B, \Bbbk^*)$, where, as usual, $\Bbbk^*$ is the group of invertible elements in $\Bbbk$. Transfer, which is a push-forward operation on HQFTs, is also discussed. It is worth noting that this construction is an example in which the possibility of varying the base B can be exploited in a way that is not so easily mirrored in the pure theory of TQFTs.

The case of an aspherical base B is then considered. As was said above, this is the main case examined in the monograph. Here, therefore, B is a $K(G, 1)$ and so, for instance, the cohomology classes used in the construction of HQFTs can be thought of as being in the group cohomology, $H^{d+1}(G, \Bbbk^*)$, rather than in $H^{d+1}(B, \Bbbk^*)$. After exploring homotopy and isotopy invariance, the next major topic introduced is that of Hermitian and unitary HQFTs. Here, if $\Bbbk$ has a ring involution, one requires that the vector spaces $\tau(X, g)$ have a bilinear Hermitian pairing. Then, if $(M, g) : (X_0, g_0) \to (X_1, g_1)$ is a B-cobordism, the opposite B-cobordism, $(-M, g)$, is assigned the linear transformation that is adjoint to $\tau(M, g)$.

Chapter II is entitled *Group algebras*, but the objects in question are not just ordinary group algebras—they generalise the usual group algebras, $\Bbbk[G]$, in several ways. These algebras will play the analogous role to that played by commutative Frobenius algebras in classical 2-dimensional topological quantum field theory (2d-TQFT). They abstract some of the structure of classical group algebras. They are G-graded and are also Frobenius algebras, but in general are noncommutative. A bit more detail will help here. A G-graded algebra over $\Bbbk$ is a direct sum of vector spaces, indexed by the elements of G,

$$L = \bigoplus_{g \in G} L_g,$$

and the associative multiplication is, of course, related to the grading, so, if $\ell_g \in L_g$ and $\ell_h \in L_h$, then $\ell_g.\ell_h \in L_{gh}$. If G is noncommutative, then so will be this multiplication, since $\ell_h.\ell_g$ will not even be in the same summand of L. Of course, multiplying something in $L_{ghg^{-1}}$ by something in L_g will again get you something in L_{gh}, and in these algebras passing a homogeneous element through another from right to left gives you something in the summand graded by the obvious conjugate of the old grade. Explicitly, this is systemised by, for each g, there being an automorphism φ_g of L that sends L_h into $L_{ghg^{-1}}$, and such that $\varphi_g(\ell_h)\ell_g = \ell_g.\ell_h$. The φ_g together give a homomorphism from G to $\mathrm{Aut}(L)$. Turaev calls this a crossed G-algebra structure, and, if combined compatibly with a Frobenius algebra structure, the term "crossed Frobenius G-algebra" is used.

Jumping ahead in the book for a moment, it is possible to see why these objects would be useful. If Z is a 2d-TQFT, then the corresponding commutative Frobenius algebra is built on the vector space $Z(S^1)$, using various cobordisms to give the algebraic structure. If τ is a 2d-HQFT based on a $K(G,1)$, there will be many B-manifold structures on S^1, given, up to equivalence, by S^1 together with an element of $\pi_1(B)$, which is, of course, G. We thus would expect to get a vector space $\tau(S^1,g)$ for each g in G, and the geometric arguments that produce the multiplication for $Z(S^1)$ would look to adapt easily to give partial "multiplications",

$$\tau(S^1,g)\otimes\tau(S^1,h)\to\tau(S^1,gh),$$

i.e., a graded algebra structure on $\bigoplus_{g\in G}\tau(S^1,g)$. The other structure adapts well, with well-known images (pairs of pants, etc.), from discussions of 2d-TQFT, but suitably labelled, giving much of the remaining structure of a crossed Frobenius G-algebra. There are one or two additional features here though. There are the φ_g's, which correspond to labelled cylinders, and these interact nicely with the other structure such as the inner product, all fitting together in a way that is quite deep, yet is also visually accessible to anyone who knows the basics of the classification of surfaces and a small amount of homotopy theory—essentially the intuitions underpinning the theory of the fundamental group. The axioms for a crossed Frobenius G-algebra then clearly mirror the topological structure precisely.

Returning to Chapter II, the next few sections deal with semisimple crossed G-algebras and their classification, and, finally, with Hermitian G-algebras.

Chapter III is devoted to giving a detailed proof of the correspondence between 2d-HQFTs over a $K(G,1)$, and crossed Frobenius G-algebras. Then in Chapter IV a subclass of crossed G-algebras, called "biangular G-algebras" is examined. These biangular G-algebras are then used to describe a state sum model on B-surfaces that gives a B-HQFT. This generalises the construction of Fukuma, Hosono, and Kawai for a 2d-TQFT; see [2].

In Chapter V, *Enumeration problems in dimension two*, the previously developed machinery of 2d-HQFTs and crossed G-algebras is applied to problems involving the enumeration of liftings of group morphisms and related topological situations. Let $q: G'\to G$ be a group epimorphism with finite kernel Γ, and consider a homomorphism, $g:\pi\to G$. A *lift* of g to G' is, as one would expect, a homomorphism $g':\pi\to G'$ such that $qg'=g$. The set of lifts of g will be denoted $\mathrm{Hom}_g(\pi,G')$. It may, of course, be empty. If g has trivial image, then, clearly, $\mathrm{Hom}_g(\pi,G')$ is just $\mathrm{Hom}(\pi,\Gamma)$. If π is $\pi_1(W)$ for W a closed connected oriented surface, the Frobenius–Mednykh formula says

$$|\mathrm{Hom}(\pi,\Gamma)| = |\Gamma|\sum_\rho(|\Gamma|/\dim\,\rho)^{-\chi(W)} = |\Gamma|\sum_\rho(|\Gamma|/\dim\,\rho)^{2d-2},$$

where d is the genus of W, and, in the summation, $\rho\in\mathrm{Irr}(\Gamma;\mathbb{C})$, the set of equivalence classes of irreducible finite-dimensional complex representations of Γ. This formula was due to Frobenius (1896) for $W=S^2$, and the general case was found by Mednykh (1978). In the context of TQFTs, it was rediscovered in the work of Dijkgraaf and Witten (1990) and by Freed and Quinn (1993), and various other related proofs have been given since. The fact that the sum is over $\mathrm{Irr}(\Gamma;\mathbb{C})$ is significant for the general case, where $g(\pi)$ need not be trivial. The size of $\mathrm{Irr}(\Gamma;\mathbb{C})$ is the number of conjugacy classes of elements of Γ and, in this guise, plays a central role in those TQFTs constructed Dijkgraaf and Witten, and Freed and Quinn. The connection between this formula in that case of trivial image and TQFTs is replaced in the general case considered in this monograph by a connection with HQFTs. We will give the statement of Turaev's beautiful generalisation of the formula, although the details of one or two of its features will be left vague as they would require too large a digression for this review to undertake.

Theorem. *Let W be a closed connected oriented surface with fundamental group π. Let $g : \pi \to G$ be a group homomorphism, and let $\Bbbk$ be an algebraically closed field of characteristic zero. Then*

$$|\operatorname{Hom}(\pi, \Gamma)| = |\Gamma| \sum (|\Gamma|/\dim \rho)^{-\chi(W)} g^*(\zeta_\rho)([W]),$$

where the summation is over those $\rho \in \operatorname{Irr}(\Gamma; \Bbbk)$ such that the stabiliser, G_ρ, of ρ contains the image $g(\pi)$ of π and $g^*(\zeta_\rho)([W])$ is the evaluation of $g^*(\zeta_\rho) \in H^2(\pi; \Bbbk^*)$ on the fundamental class of W.

The cohomology class $\zeta_\rho \in H^2(G_\rho; \Bbbk^*)$ is dependent only on the equivalence class of ρ.

The result and the surrounding theory is explored in depth throughout this chapter. It is noted that Gareth Jones (1995) gave a purely algebraic proof of the Frobenius–Mednykh formula, and it would be interesting to search for an algebraic proof of this generalisation. (This may be seen as another instance of the usefulness of HQFT techniques to get around assumptions needed in the use of TQFTs.) Some very interesting questions relating to non-abelian cohomology and the enumeration of fibre bundles on surfaces are discussed towards the end of the chapter.

Chapter VI, *Crossed G-categories and invariants of links*, starts on the investigation of 3d-HQFTs, still with the base, B, being a $K(G, 1)$. The connection between 3d-TQFTs of various types and monoidal categories is well established. One approach uses the theory of links in S^3 and it is that approach that is used to generalise those results here. A *G-category* will be a $\Bbbk$-additive monoidal category $\mathcal{C}$ with left duality that splits as a disjoint union of subcategories $\mathcal{C}_g$ $g \in G$ such that

(i) the unit object $\mathbb{1} \in \mathcal{C}_1$, and if $U \in \mathcal{C}_g$, $V \in \mathcal{C}_h$, then $U \otimes V \in \mathcal{C}_{gh}$,

and

(ii) if $U \in \mathcal{C}_g$, its dual U^* is in $\mathcal{C}_{g^{-1}}$.

(This is clearly a "categorification" of a graded G-algebra.) The process of passing to "crossed" versions of G-categories is then a fairly easy extension of the case of algebras. Crossed G-categories come in various flavours—braided, ribbon, etc.—as the usual theory would suggest, and just as in that theory there is a strong connection with links and tangles. In this G-graded context, these will be coloured using G and related invariants of G-coloured tangles. G-coloured graphs, etc., are explored in the second half of this chapter.

In Chapter VII, modular G-categories are introduced. Modular categories, in this sense, were introduced by Turaev in his earlier book [9]. They play an important role in the study of TQFTs, conformal theories, and quantum groups. Their definition is a bit too long to give in detail here, but they are monoidal categories that are abelian with finitely many isomorphism classes of simple objects and for which the so-called S-matrix is invertible. This structure adapts well to a G-graded context, and Turaev shows here that any modular crossed G-category gives rise to a 3d-HQFT. This HQFT then provides numerical invariants of closed oriented 3d G-manifolds.

In Chapter VIII, a G-graded version of quasi-triangular Hopf (co)algebras is introduced, and it is shown that they give rise to crossed G-categories in much the same way that quasi-triangular Hopf algebras have braided representation categories. This chapter also contains descriptions of algebraic constructions of crossed G-algebras.

The book ends with some appendices, including one by Michael Müger on braided crossed G-categories, and two by Alexis Virelizier on Hopf G-(co)algebras and their applications.

Since the initial theory was described in the preprints in 1999 and 2000, there have been quite a few studies into further aspects, extensions, and applications of the theory. Some have tried to go away from the restriction that B is a $K(G,1)$ that is considered for most of this book; others have investigated links with other areas. We cannot give an exhaustive list of these, nor can we do more than comment briefly on those that are mentioned. (The monograph has a brief appendix looking at various other related aspects.)

- Kaufmann, in a series of articles (for example [3]), considers the Frobenius algebras that arise from singularities. If there is a group of symmetries of that singularity, the result is a (slight generalisation of) a crossed G-algebra. These have been developed by him to study the quantisation of Frobenius algebras and orbifold models for conformal field theories.
- The highly influential paper of Moore and Segal [6] takes "Turaev algebras", that is crossed G-algebras, as a central part of their discussion of open-closed field theories. In particular, they look at the algebra of invariants of a crossed G-algebra.
- Turaev, with the reviewer [7], looked at the situation for 2d-HQFTs in which the base, B, was the classifying space of a crossed module, C, and hence a general 2-type. The interpretation of the HQFT as being a TQFT of manifolds with fixed principal G-bundle, has here to be replaced with a more subtle one of manifolds with a principal C-2-bundle. There are still a lot of unanswered question in this area.
- Lurie in [5] introduced a notion of extended TQFT for manifolds with an (X,ζ)-structure—an idea which is closely related to that of B-manifolds.

The theoretical foundations in all of these show connections far and wide into classification of various types of field theories, types of stack, and the possibility of new manifold invariants, but this is not always developed, and the connections with Turaev's HQFTs are not always explicitly given, nor are they exploited.

Having finished the book, I am left with a feeling almost of frustration, similar to finishing the first of a series of thrillers where the characters have been introduced and some "adventures" have happened to them, yet there are so many unanswered questions that come to mind. The theory is powerful, interesting, challenging, etc., but, like many new areas of mathematics, it leaves so much unsaid and unknown, and there are situations that were crying out for further development.

It is a very enjoyable area to study. It needs some prior knowledge of TQFTs, and anyone wanting to study the area who does not have some reasonably good idea of that subject would be well advised to read up on it before plunging into this book. It would make an excellent monograph on which to base a joint study seminar at the graduate or postdoc level. Such a study would be likely to raise new questions and result in new insights and clarifications of the ramifications of homotopy quantum field theories.

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Timothy Porter

University of Bangor, United Kingdom

E-mail address: t.porter@bangor.ac.uk