

WHY SHOULD THE LITTLEWOOD–RICHARDSON RULE BE TRUE?

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ABSTRACT. We give a proof of the Littlewood-Richardson Rule for describing tensor products of irreducible finite-dimensional representations of GL_n . The core of the argument uses classical invariant theory, especially (GL_n, GL_m) -duality. Both of the main conditions (semistandard condition, lattice permutation/Yamanouchi word condition) placed on the tableaux used to define Littlewood-Richardson coefficients have natural interpretations in the argument.

1. INTRODUCTION

The Littlewood-Richardson (LR) Rule is a central result in the representation theory of compact Lie groups, *mutatis mutandis*, of reductive complex algebraic groups ([Fu1], [LR], [Ma], [Sa], [vL], [Ze]). It is also a topic of intense study in combinatorics, especially the theory of symmetric functions ([Ma], [Su], [Sch]), and it has intimate connections with the topology of Grassmann varieties, and thereby the theory of vector bundles and K -theory ([Hus]). First stated by Littlewood and Richardson in 1934 ([LR]), it was fully proven only in the late 1970s ([Sch], [T1], [T2], [Ma]) by means of refined study of the combinatorics of partitions and symmetric functions. More recently, several authors have given relatively short proofs ([Stm], [Ze], [ReS]).

Besides the long wait for a full proof, another notable feature of the LR Rule is the curious way in which it is usually formulated. It is stated in terms of certain combinatorial objects called tableaux (described carefully below). It asserts that certain numbers of interest can be counted by tableaux satisfying two types of conditions. (We will call these special tableaux “LR tableaux”.) One condition is *semistandardness*. Semistandardness is a condition that arises in a straightforward way as part of the work of giving concrete combinatorial descriptions of representations of the general linear group. The other condition is formulated in various ways, in terms of *lattice permutations* ([Ma]) or as the *Yamanouchi word condition* ([Fu1]). The role this condition plays is not a priori transparent.

In this paper we offer another proof of the LR Rule. This proof is not shorter than some currently in the literature, but we hope that it has some other virtues.

Received by the editors March 30, 2009, and, in revised form, February 14, 2011.

2000 *Mathematics Subject Classification*. Primary 20G05; Secondary 05E15.

Key words and phrases. Littlewood-Richardson Rule, Pieri Rule, GL_n tensor product algebra, (GL_n, GL_m) -duality.

The second named author is partially supported by NUS grant R-146-000-110-112.

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Whereas almost all proofs noted above are based on combinatorics, this one uses representation theory in substantial ways. Moreover, it is a refinement of the original partial proof of Littlewood and Richardson, so that it shows how their argument can be completed by appropriate use of representation theory. We also hope that it gives some insight into why the standard statement of the LR Rule is reasonable. More precisely, our argument puts the semistandardness condition and the other condition on essentially equal footings. It uses representation theory to argue that the combination of the two conditions is what you might hope would characterize the phenomenon described by the LR tableaux. Also, it interprets the lattice permutation/Yamanouchi word condition in terms of (GL_n, GL_m) -duality. See §7 for details.

Here is a preview of the rest of the paper. In §2, we briefly review the varied applications of the LR Rule. In §3, we review some basic facts in the representation theory of GL_n . In §4, we state the LR Rule in its standard form. In §5, we discuss it in the light of representation theory and provide an alternative interpretation of the conditions defining LR tableaux. We hope that this discussion makes the LR Rule seem natural and inevitable. In §6, we recall the arguments of Littlewood and Richardson concerning a special case of the rule, and put these in a representation-theoretic setting. This argument has a strong combinatorial component and is closely connected to the Catalan numbers. It is also suggestive of the path model descriptions of tensor products and branching rules developed by Littelmann ([Lim]). Finally in §7, we show how the constructions of [HTW1], [HTW3], and [HL], of explicit highest weight vectors in tensor products, allow one to establish the general case of the LR Rule on the basis of the special case treated by Littlewood and Richardson.

A key tool in the arguments of §7 is the theory of of SAGBI bases ([RuS]), especially the use of leading monomials to analyze the structure of various algebras. The backdrop for the calculations of §7 is the standard monomial theory of Hodge. This theory has undergone intensive study and now can be proven more simply, and interpreted more conceptually, than when it was first established. Although it is not strictly needed for the proof of the LR Rule, at the end of §7.2, we sketch some of the recent history of standard monomial theory.

2. APPEARANCES OF THE LITTLEWOOD-RICHARDSON RULE

We offer here a brief overview of some of the occurrences and applications of Littlewood-Richardson coefficients. These topics are discussed in much greater detail in several works of Fulton ([Fu1], [Fu2]).

2.1. Representation theory. Over the years since its discovery, it has become understood that the LR Rule unifies a large range of phenomena in representation theory and the geometry of Lie groups and homogeneous spaces.

In the original paper ([LR]), the LR Rule was formulated to describe how to decompose the tensor product of two irreducible representations of the unitary group (equivalently, two irreducible “regular” or “rational” representations of the complex general linear group) into a sum of irreducible representations. It is now understood that LR coefficients can be used to describe branching rules for all families of classical symmetric pairs ([HTW2]).

Given a group G and a subgroup H , the *branching rule* from G to H is the description of how irreducible representations of G decompose into irreducible H -subrepresentations. If G and hence H is compact, then any representation π of H will decompose into a direct sum of irreducible representations, and the only information one needs to describe π up to isomorphism is the multiplicity $m(\pi, \rho)$ with which each irreducible representation ρ of H appears in π . Thus, knowledge of the multiplicities $m(\sigma|_H, \rho) = m(\sigma, \rho)$ of the restriction to H of each irreducible representation σ of G is sufficient to provide at least a numerical form of the branching rule from G to H . The $m(\sigma, \rho)$ are called *branching multiplicities* from G to H (or for the pair (G, H)).

Given a (finite or compact) group G , a tensor product $\sigma \otimes \tau$ of two irreducible representations of G may be regarded as an irreducible representation of the product $G \times G$ of G with itself. This is sometimes called the *outer* tensor product of σ and τ , whereas the same tensor product, thought of as a representation of the original group G , is called the *inner* tensor product. The relation between the outer tensor product and the inner tensor product may be thought of in terms of restricting a representation to a subgroup. Indeed, if we embed G into $G \times G$ diagonally, by the mapping $\Delta : g \rightarrow (g, g)$ for $g \in G$, then the inner tensor product of σ and τ results from restricting the outer tensor product to $\Delta(G)$.

The group $G \times G$ has an obvious involution (automorphism of order 2) given by $\gamma : (g_1, g_2) \rightarrow (g_2, g_1)$. It is easily checked that $\Delta(G)$ is the set of fixed points of γ . This means that $\Delta(G)$ is a *symmetric subgroup* of $G \times G$. In general, a subgroup $K \subset G$ is called a symmetric subgroup if K consists of the fixed points of an involution (automorphism of order 2) of G . A pair (G, K) where K is a symmetric subgroup of G is called a *symmetric pair*.

A *classical algebraic group* is a member of one of the three families of the general (or special) linear groups, orthogonal (or special orthogonal, or spin), or symplectic groups; or products of these. The symmetric pairs (G, K) of classical Lie groups can be put into ten infinite families ([HTW1]) comprising four general types: diagonal subgroups, direct sum decompositions, isometry groups, and stabilizers of polarizations. The general linear group embedded diagonally in its product with itself is one of these ten families, and the LR Rule is exactly the branching law for this family of symmetric pairs. The multiplicity of a representation ν in the tensor product of two other irreducible representations λ and μ is denoted $c'_{\lambda, \mu}$. The numbers $c'_{\lambda, \mu}$ are called *Littlewood-Richardson (LR) coefficients*.

It turns out that, at least under some technical restrictions, the branching rules for all ten families of classical symmetric pairs (G, K) can be described in terms of LR coefficients. A substantial literature is devoted to showing this, from work of Littlewood in the early 1940s (the Littlewood restriction rule [Liw1]), continuing through the 1990s ([BKW], [Liw2], [Liw3], [Ki1], [Ki2], [Ki3], [Ki4], [Ko], [KoT]). For a review of the work up to about 1990, see [Su], and for a unified treatment, see [HTW2].

We note that, by Frobenius Reciprocity ([GW]), knowing the branching rule from G to a subgroup K is equivalent to being able to decompose the induced representation $\text{Ind}_K^G \sigma$ from any representation σ of K . When G and K are compact Lie groups, this amounts to the spectral analysis of sections of the homogeneous vector bundle defined by σ over the homogeneous space G/K (which is a compact symmetric space when (G, K) is a symmetric pair).

The application of LR coefficients in representation theory is not limited to Lie groups. It turns out that the restriction of irreducible representations of a symmetric group S_n on n letters to the subgroup $S_m \times S_{n-m}$ is also described by LR coefficients (i.e., the branching rule from S_n to $S_m \times S_{n-m}$). This appearance of the LR coefficients in the representation theory of the symmetric group is related to the beautiful *Schur duality* ([W2], [Ho2], [GW] etc.), which gives a natural correspondence between representations of the S_n on the one hand and $GL_m(\mathbb{C})$ on the other.

2.2. Combinatorics. The influence of the LR Rule is not limited to representation theory. The search for its proof greatly stimulated combinatorics, specifically, the combinatorics of Young diagrams and tableaux. It is in this context, via the tools of the Robinson-Schensted-Knuth correspondence and the *jeu de taquin* of Schützenberger that many of the proofs in the literature are given ([Sch], [T1], [T2], [Ma]). This proof computes (generalizations of) the LR coefficients as the number of ways that a skew Young diagram with given content can be “rectified” to a given standard tableau. Several other interpretations of the LR coefficients in terms of tableaux or skew tableaux, including the “pictures” interpretation of Zelevinsky ([Ze]), have also been given. See [Fu1] for further discussion.

The original motivation for the LR Rule, and indeed the main focus of [LR], is the theory of symmetric functions. As discussed in Macdonald’s book ([Ma]), there are many bases for the ring of symmetric functions, each valuable for a different purpose, and a substantial portion of the theory is devoted to mediating between these bases. One basis that was of considerable interest was the basis of S -functions (or Schur functions). This basis was studied during the 19th century, but has been named for Schur since his work identified it with the characters of $GL_n(\mathbb{C})$ (equivalently, of the unitary group $U(n)$) in an appropriate coordinate system. (This amounts to the Weyl Character Formula ([W2]) for the unitary group.) The LR coefficients are the structure constants for the ring structure of the symmetric functions with respect to the basis of S -functions. Since the character of the tensor product of representations is the product of the characters of the factors, the identification of S -functions with characters makes the link between the representation theoretic definition and the symmetric function definition. However, the symmetric function interpretation stands on its own within that area of study.

2.3. Geometry. The LR coefficients show up in the topology of Grassmann varieties (and are thereby relevant to the theory of vector bundles) ([Le], [Fu1], [Fu2]). The Grassmannian $G_{n,d}(\mathbb{C}) = G_{n,d}$ is the set of d -dimensional planes in an n -dimensional complex vector space. It can be decomposed into a union of $\binom{n}{d}$ cells of even dimension. The cells are known as *Bruhat cells*, and their closures are (possibly singular) subvarieties of $G_{n,d}$ known as *Schubert varieties*. They are so named because they are basic to a rigorous approach to enumerative geometry pioneered by Schubert in the 19th century ([Sch]).

We will pause to describe the Schubert varieties. Recall that a *flag* in \mathbb{C}^n is a sequence of subspaces U_i that are *nested*, in the sense that $U_i \subset U_{i+1}$. A flag is *complete* if it contains one subspace of every dimension: $\dim U_i = i$ for $0 \leq i \leq n$. Given a (an ordered) basis $\mathcal{B} = \{\vec{\mathbf{b}}_i : 1 \leq i \leq n\}$ for \mathbb{C}^n , we can define a complete flag $\mathcal{F}_{\mathcal{B}}$ by letting $U_{\mathcal{B},i} = U_i$ be the span of the basis first i elements: $\vec{\mathbf{b}}_a$ for $1 \leq a \leq i$. We call this the *upper flag* associated to \mathcal{B} .

If we use the standard Hermitian inner product on \mathbb{C}^n , then a version of the Gram-Schmidt procedure allows us attach to a complete flag \mathcal{F} an orthonormal basis $\mathcal{B} = \{\vec{\mathbf{u}}_i\}$ such that $\mathcal{F} = \mathcal{F}_{\mathcal{B}}$ is the upper flag attached to this basis. More canonically, we can attach to \mathcal{F} the collection of lines $L_i = U_i \cap (U_{i-1}^\perp)$. Here X^\perp indicates the orthogonal complement with respect to the Hermitian form of the subspace $X \subset \mathbb{C}^n$. Then $\vec{\mathbf{u}}_i$ can be any unit vector in L_i . Thus, the elements of \mathcal{B} are specified only up to multiplication by scalars of absolute value 1.

Fix a complete flag $\mathcal{F} = \{U_i\}$ in \mathbb{C}^n . Given a subspace $V \subset \mathbb{C}^n$, the intersections $V \cap U_j$ will form a nested sequence of subspaces of V , and it is easy to see that $\dim(V \cap U_j) \leq \dim(V \cap U_{j-1}) + 1$. Thus, except for the facts that there is redundancy and some of the $V \cap U_j$ are equal to each other, these subspaces form a complete flag in V . We will call the sequence of j such that $V \cap U_j \neq V \cap U_{j-1}$ the *jump sequence* of V .

Denote the jump sequence of V relative to \mathcal{F} by $J_{\mathcal{F},V} = J_V$. Thus, J_V is a strictly increasing sequence of integers, which we will list in a column of length $d = \dim V$:

$$(2.1) \quad J_{\mathcal{F},V} = J_V = J = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ \vdots \\ j_d \end{bmatrix}.$$

It is easy to convince oneself that any collection J of d distinct whole numbers $j_a, 1 \leq a \leq d$, from 1 to n , arise as the jump sequence of some d -dimensional subspace V . Indeed, if \mathcal{F} is the upper flag for the basis $\mathcal{B} = \{\vec{\mathbf{b}}_i\}$ and J is a list of d indices from 1 to n , just take $V_{\mathcal{B},J} = V_J$ to be the span the basis vectors $\vec{\mathbf{b}}_{j_a}$ for j_a in J . Then it is obvious that the jump sequence of V_J is exactly J . Thus, there are exactly $\binom{n}{d}$ possibilities for the jump sequence of V . Denote the set of d -dimensional subspaces V with jump sequence equal to J by $B_{\mathcal{F},J} = B_J$.

Let $\mathbf{B}_{\mathcal{F}}$ be the subgroup of $\text{GL}_n(\mathbb{C})$ that stabilizes the spaces U_j of the complete flag \mathcal{F} . It is clear that if V is a subspace of \mathbb{C}^n and g is in $\mathbf{B}_{\mathcal{F}}$, then $g(V)$ has the same jump sequence with respect to \mathcal{F} as V does. Thus, the Bruhat cells $B_{\mathcal{F},J}$ are invariant under $\mathbf{B}_{\mathcal{F}}$. It is well known and follows from reduction theory in elementary linear algebra that in fact $B_{\mathcal{F},J}$ is just the $\mathbf{B}_{\mathcal{F}}$ orbit in $G_{n,d}$ of the space V_J :

$$(2.2) \quad B_{\mathcal{F},J} = B_J = \{V : J_{\mathcal{F},V} = J\} = \mathbf{B}_{\mathcal{F}}(V_J).$$

Reduction theory also reveals that this set can be parametrized by a complex vector space of dimension

$$(2.3) \quad \sum_{a=1}^d (j_a - a) = \left(\sum_{a=1}^d j_a \right) - \frac{d(d+1)}{2}.$$

Thus, topologically, B_J is a cell of dimension $2 \left(\left(\sum_{a=1}^d j_a \right) - \frac{d(d+1)}{2} \right)$.

The closure $\overline{B}_{\mathcal{F},J} = \Omega_{\mathcal{F},J}$ of a given Bruhat cell B_J is called a *Schubert variety*. It is a union of the original cell together with other Bruhat cells. Thus, closure of Bruhat cells induces a natural partial ordering, often referred to as the *Bruhat order*, on the set of jump sequences: we will say that, for jump sequences J and K , that $K \leq J$ if and only if $B_K \subset \overline{B}_J$. To describe this partial order, we observe

that, from the definition of jump sequence, it is clear that if J_V is as in equation (2.1), then $\dim(V \cap U_k) = a$ for $j_a \leq k < j_{a+1}$. Since the dimension of intersection of a variable space V with a fixed space U_k can only increase as V approaches a limit, it follows that, if V' is in \overline{B}_J , then each jump of V' must occur not later than the corresponding jump of B_J . It is not hard to check that this condition is also sufficient. Thus,

$$(2.4) \quad K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_d \end{bmatrix} \leq \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ \vdots \\ j_d \end{bmatrix} = J \quad \text{if and only if } k_a \leq j_a$$

for $1 \leq a \leq d$. Thus, the Bruhat order is compatible with the natural partial ordering on the lattice \mathbb{Z}^n of integer sequences.

Since the Bruhat cells are all of even dimension, it follows from the general machinery of algebraic topology that the homology classes $[\Omega_{\mathcal{F},J}]$ defined by the Schubert varieties $\Omega_{\mathcal{F},J}$ form a basis for the integral homology of $G_{n,d}$. Since $GL_n(\mathbb{C})$ acts transitively on the set of complete flags, and since it is a connected group, it follows that the homology class $[\Omega_{\mathcal{F},J}]$ is independent of the complete flag \mathcal{F} . Thus it is entirely appropriate to suppress the dependence of the homology class on \mathcal{F} , and write

$$(2.5) \quad [\Omega_{\mathcal{F},J}] = [\Omega_J],$$

and the $[\Omega_J]$ are then an integral basis for the homology of $G_{n,d}$.

There is a natural duality on Bruhat cells and Schubert cells. Given the ordered basis $\mathcal{B} = \{\vec{\mathbf{b}}_i : 1 \leq i \leq n\}$, we define the *opposite basis* to be the basis consisting of the same vectors, but in the reverse order:

$$(2.6) \quad \mathcal{B}^{\text{opp}} = \{\vec{\mathbf{b}}_i^{\text{opp}} = \vec{\mathbf{b}}_{n+1-i} : 1 \leq i \leq n\}.$$

We then define the *lower* or *opposite* flag $\mathcal{F}_{\mathcal{B}}^{\text{opp}} = \mathcal{F}_{\mathcal{B}^{\text{opp}}}$ associated to \mathcal{B} as the usual upper flag associated to \mathcal{B}^{opp} . Thus U_i^{opp} is the span of $\vec{\mathbf{b}}_a$ for $n+1-i \leq a \leq n$. If \mathcal{B} is an orthonormal basis, then we can also write $U_i^{\text{opp}} = U_{n-i}^\perp$, where the U_i are the spaces of the upper flag attached to \mathcal{B} .

Let $\mathbf{B}_{\mathcal{F}}^{\text{opp}}$ be the subgroup of $GL_n(\mathbb{C})$ stabilizing the lower flag $\mathcal{F}_{\mathcal{B}}^{\text{opp}}$ attached to \mathcal{B} . If $V_{\mathcal{B},J}$ is, as above, the span of the basis elements $\vec{\mathbf{b}}_j$ with j belonging to J , then with respect to \mathcal{B}^{opp} we have

$$(2.7) \quad V_{\mathcal{B},J} = V_{\mathcal{B}^{\text{opp}},J^{\text{opp}}},$$

where

$$(2.8) \quad J^{\text{opp}} = \begin{bmatrix} n+1-j_d \\ n+1-j_{d-1} \\ n+1-j_{d-2} \\ \vdots \\ n+1-j_1 \end{bmatrix}.$$

It is then easy to see that the intersection $B_{\mathcal{F},J} \cap B_{\mathcal{F}^{\text{opp}},J^{\text{opp}}}$ of the Bruhat cells for $\mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathcal{B}}^{\text{opp}}$ generated by $V_{\mathcal{B},J}$ consists exactly in the one space (point in $G_{n,d}$)

$V_{\mathcal{B},J}$ and this will remain true for the associated Schubert varieties:

$$(2.9) \quad \Omega_{\mathcal{F}_{\mathcal{B}},J} \cap \Omega_{\mathcal{F}_{\mathcal{B}}^{\text{opp}},J^{\text{opp}}} = \{V_{\mathcal{B},J}\}.$$

Moreover the dimension formula (2.3) shows that $\Omega_{\mathcal{F}_{\mathcal{B}},J}$ and $\Omega_{\mathcal{F}_{\mathcal{B}}^{\text{opp}},J^{\text{opp}}}$ have complementary dimensions, so that

$$(2.10) \quad \dim \Omega_{\mathcal{F}_{\mathcal{B}},J} + \dim \Omega_{\mathcal{F}_{\mathcal{B}}^{\text{opp}},J^{\text{opp}}} = d(n - d) = \dim G_{n,d}.$$

Although the parametrization of Bruhat cells by their jump sequences is direct and convenient, another parametrization is frequently used is by partitions or Young diagrams. A partition α is specified by a weakly decreasing sequence of non-negative integers. Since the entries j_a of the column J recording a jump sequence are strictly increasing, we can use them to define a partition α_J with row lengths

$$(2.11) \quad \alpha_J = \begin{bmatrix} j_d - d \\ j_{d-1} - (d - 1) \\ \vdots \\ j_1 - 1 \end{bmatrix}.$$

The Young diagram D_{α_J} associated to α_J has at most d non-zero rows, and rows of length at most $n - d$. (See §4 for an introduction to partitions and diagrams.) Thus, it fits inside the rectangle $R_{d,n-d}$ consisting of d rows of length $n - d$. It is not hard to check that all such diagrams arise, and that

$$(2.12) \quad J \leftrightarrow \alpha_J \leftrightarrow D_{\alpha_J}$$

is a bijection between subsets of d of the integers from 1 to n and partitions (or their diagrams) that fit in $R_{d,n-d}$. Also, it is clear from the definition of D_{α_J} that its size (the total number of boxes it contains) is $|D_{\alpha_J}| = \sum_{a=1}^d (j_a - a)$. As we have noted, this is the dimension of the Bruhat cell B_J :

$$(2.13) \quad |D_{\alpha_J}| = \dim B_J.$$

Parametrizing Bruhat cells by diagrams also fits well with the duality described above. Given a diagram D in the rectangle $R_{d,n-d}$, the complement $R_{d,n-d} - D$, when rotated by 180° , is again a diagram, which we will denote by D^{opp} . The row lengths of D^{opp} give the partition

$$(2.14) \quad \alpha^{\text{opp}} = \alpha_{D^{\text{opp}}} = \begin{bmatrix} n - d - (j_1 - 1) \\ n - d - (j_2 - 2) \\ \vdots \\ n - d - j_d - d \end{bmatrix} = \alpha_{J^{\text{opp}}}.$$

The prevailing convention is to label the homology class of a Schubert cell by the partition of this complementary diagram. Thus, for a partition α , we set

$$(2.15) \quad \omega_\alpha = [\Omega_{J_{\alpha^{\text{opp}}}}].$$

Thus, the size of α gives the codimension, rather than the dimension, of ω_α . Also, inclusion of Schubert varieties corresponds to reverse inclusion of diagrams.

If we now dualize again and define σ_α to be the cohomology class that is (Poincaré) dual to the homology class ω_α , then the σ_α are a basis for the cohomology of $G_{n,d}$. Also, note that $\#(\alpha)$ now tells us the degree of σ_α . According

to [Le], the cup product on cohomology can be expressed with respect to this basis using the Littlewood-Richardson coefficients. For partitions α and β , we have

$$(2.16) \quad \sigma_\alpha \cdot \sigma_\beta = \sum_\gamma c_{\alpha,\beta}^\gamma \sigma_\gamma.$$

Here the sum is over all partitions γ such that $\#(\gamma) = \#(\alpha) + \#(\beta)$. Since all the classes ω_α are represented by complex subvarieties of $G_{n,d}$, equation (2.16) may be interpreted in terms of intersection theory ([Fu3]). This says that for partitions α , β , and γ with $\#(\gamma) = \#(\alpha) + \#(\beta)$, and any three flags \mathcal{F} , \mathcal{F}' and \mathcal{F}'' , we have that the cardinality of the intersection

$$(2.17) \quad \left(\Omega_{\mathcal{F}', J_{\alpha}^{\text{opp}}} \cap \Omega_{\mathcal{F}'', J_{\beta}^{\text{opp}}} \cap \Omega_{\mathcal{F}, J_\gamma} \right) \geq c_{\alpha,\beta}^\gamma.$$

In particular, the intersection is guaranteed to be non-zero exactly when the LR coefficient $c_{\alpha,\beta}^\gamma$ is non-zero. For generic choices of the flags, the intersection will be exactly equal to the Littlewood-Richardson coefficient number of points.

2.4. Sums of Hermitian matrices. The involvement of the LR coefficients in describing intersections of Schubert varieties has also implicated them in the solution ([Kl], [KT1], [KT2], [KTW], see also [Fu2]) to the Horn Conjecture, which was a proposal for a precise answer to a long-standing problem in spectral theory: to describe the possible eigenvalues of a sum of two Hermitian $n \times n$ matrices with specified eigenvalues. Weyl ([W1]), using his *min-max* argument, established some inequalities between the eigenvalues of A , B and $A + B$ for Hermitian matrices A and B . We describe his result.

Let $\lambda_j(T)$, $1 \leq j \leq n$, be the eigenvalues of the Hermitian matrix T , arranged in order from largest to smallest. Thus, $\lambda_j(T)$ is the j th largest eigenvalue of T . In this notation, for Hermitian matrices A and B , Weyl found estimates

$$(2.18) \quad \lambda_{a+b-1}(A + B) \leq \lambda_a(A) + \lambda_b(B)$$

for any positive integers a and b . Thus, if $a = 1$ and $b = 2$, this says that $\lambda_2(A+B) \leq \lambda_1(A) + \lambda_2(B)$.

The key to Weyl’s argument involved looking at the inner products $(T\vec{v}, \vec{v})$, where $(\ , \)$ denotes the standard Hermitian inner product on \mathbb{C}^n and \vec{v} is a unit vector in \mathbb{C}^n . If $\mathcal{B}_T = \{\vec{u}_j\}$ is an orthonormal eigenbasis for T , with $T\vec{u}_j = \lambda_j(T)\vec{u}_j$, then if we express $\vec{v} = \sum_i c_i \vec{u}_i$ as a linear combination of the \vec{u}_j , we can write $(T\vec{v}, \vec{v}) = \sum_i \lambda_i(T) |c_i|^2$.

Now suppose that \vec{v} belongs to the subspace $U_{T,j}$ spanned by the \vec{u}_i for $i \leq j$. Then the sum expressing $(T\vec{v}, \vec{v})$ terminates at $i = j$:

$$(2.19) \quad (T\vec{v}, \vec{v}) = \sum_{i \leq j} \lambda_i(T) |c_i|^2 \geq \lambda_j(T) \sum_{i \leq j} |c_i|^2 = \lambda_j(T) (\vec{v}, \vec{v}) = \lambda_j(T),$$

since \vec{v} is a unit vector. A similar argument shows that if \vec{v} belongs to $U_{T,j}^\perp$, the orthogonal complement of $U_{T,j}$, then $(T\vec{v}, \vec{v}) \leq \lambda_{j+1}(T)$.

In the context of comparing the spectrum of $A + B$ to those of A and B , suppose that \vec{v} is a unit vector belonging to $U_{(A+B),\ell} \cap U_{A,j}^\perp \cap U_{B,k}^\perp$. Then we can say that

$$\lambda_\ell(A + B) \leq ((A + B)\vec{v}, \vec{v}) = (A\vec{v}, \vec{v}) + (B\vec{v}, \vec{v}) \leq \lambda_{j+1}(A) + \lambda_{k+1}(B).$$

Since $\dim U_{T,i} = i$, basic linear algebra tells us that the dimension of the intersection must be at least $\ell + (n - j) + (n - k) - 2n = \ell - (j + k)$. This will be positive if $\ell > j + k$, from which inequality (2.18) follows.

Weyl’s inequalities give useful information. In fact, for $n = 2$, they, along with the obvious conditions $\lambda_1(A + B) + \lambda_2(A + B) = \text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) = \lambda_1(A) + \lambda_2(A) + \lambda_1(B) + \lambda_2(B)$, completely determine the possibilities for the spectrum of $A + B$ in terms of the spectra of A and B . However, already for 3×3 matrices, they do not give the whole story.

One might, therefore, ask, what other inequalities might hold between eigenvalues of A , B and $A + B$? The Horn Conjecture ([Horn]) proposed a solution, which was verified in the late 1990s, through a combination of several authors ([Kl], [KT1], [KTW]). The long article [Fu2] is devoted to an exposition of these results. In some sense, the key to the solution requires just a slight reformulation of Weyl’s results. However, arriving at the point of view that allowed this reformulation took over 60 years! ([W1], [Jo]) (And another 15 years to enter the published literature [To], [HR].)

The inner product $(A\vec{v}, \vec{v})$, for a unit vector \vec{v} , can also be expressed as

$$(2.20) \quad (A\vec{v}, \vec{v}) = \text{tr}(P_{\vec{v}}AP_{\vec{v}}) = \text{tr}(P_{\vec{v}}A),$$

where $\text{tr}(T)$ denotes the trace of the matrix T , and $P_{\vec{v}}$ is the orthogonal projection to the line $\mathbb{C}\vec{v}$,

$$(2.21) \quad P_{\vec{v}}(\vec{w}) = \frac{(\vec{w}, \vec{v})}{(\vec{v}, \vec{v})}\vec{v}$$

for a vector \vec{w} in \mathbb{C}^n . If $\mathbb{C}\vec{v} = L$ denotes the line spanned by \vec{v} , then we will also write $P_{\vec{v}} = P_L$. With this notation, the inequalities noted above on the eigenvalues of A can be restated as

$$(2.22) \quad L \subset U_{A,j} \quad \Rightarrow \quad \text{tr}(P_L A) \geq \lambda_j(A)$$

and

$$(2.23) \quad L \subset (U_{A,j})^\perp \quad \Rightarrow \quad \text{tr}(P_L A) \leq \lambda_{j+1}(A).$$

Now, instead of a unit vector \vec{v} , we look for a line L such that

$$L \subset U_{(A+B),\ell} \cap (U_{A,j})^\perp \cap (U_{B,k})^\perp.$$

If such an L exists, we can assert that

$$\lambda_\ell(A + B) \leq \text{tr}(P_L(A + B)) = \text{tr}(P_L A) + \text{tr}(P_L B) \leq \lambda_{j+1}(A) + \lambda_{k+1}(B).$$

The same dimension considerations as in the argument above give a rederivation of the inequality (2.18).

Reviewing the discussion above in the context of §2.3, we can recognize a connection to the geometry of Grassmann varieties. Observe that the spaces $U_{A,j}$ define a complete flag in \mathbb{C}^n . It is the upper flag (see §2.3) associated to an eigenbasis for A , ordered by decreasing size of the A -eigenvalue. Let us denote this flag by \mathcal{F}_A . Note also that the orthogonal complements $(U_{A,j})^\perp$ form the opposite flag $\mathcal{F}_A^{\text{opp}}$. Moreover, a condition that $L \subset U_{A,j}$ or $L \subset (U_{A,j})^\perp$ amounts to a condition that the point in $G_{n,1} = \mathbb{P}^{n-1}(\mathbb{C})$ represented by L belongs to a certain Schubert variety attached to the flag \mathcal{F}_A or the flag $\mathcal{F}_A^{\text{opp}}$. Finally, the condition yielding Weyl’s inequalities is seen as just the condition that certain triples of Schubert varieties in $G_{n,1}$ must intersect. (In the case of $G_{n,1}$, this is a well-known fact of linear algebra and no cohomology need be invoked. On the other hand, the cohomology of projective space is well understood and very simple, so it would not be delicate to use the cohomological criterion.)

These remarks suggest that the argument above is not restricted to lines, but can be extended to apply to subspaces of any dimension. Consider a subspace U of dimension d . Let P_U denote orthogonal projection to U . Note that, if $U = U' \oplus U''$ is an orthogonal direct sum decomposition of U , then $P_U = P_{U'} + P_{U''}$. Continuing this, we can see that if $U = \bigoplus_i L_i$ is an orthogonal direct decomposition of U into lines, then $P_U = \sum_i P_{L_i}$.

For the d -dimensional subspace V of \mathbb{C}^n , let $J_V = J$ be the jump sequence of V with respect to the flag \mathcal{F}_A , as described in §2.3. It is not hard to argue that we can find an orthonormal basis $\{\vec{y}_b\}$ for V , such that \vec{y}_b belongs to U_{A,j_b} . Using this, and the decomposition of P_V into projections corresponding to lines, as in the previous paragraph, we can see that the argument for inequality (2.19) extends to give

$$(2.24) \quad \text{tr}(P_V A) \geq \sum_{b=1}^d \lambda_{j_b}(A).$$

Similarly, if $J_V^{\text{opp}} = J^{\text{opp}}$ is the jump sequence of V with respect to the opposite flag $\mathcal{F}_A^{\text{opp}}$ defined by the spaces $U_{A,n-j}^\perp$, then we have inequalities in the opposite direction,

$$\text{tr}(P_V A) \leq \sum_{c=1}^d \lambda_{n-j_c^{\text{opp}}+1}(A).$$

We note that the eigenvalue sums in these equations can also be expressed as traces. Indeed,

$$(2.25) \quad \sum_{b=1}^d \lambda_{j_b}(A) = \text{tr}(P_{V_{J,A}} A) = \text{tr}_J(A),$$

where $V_{J,A}$ is the span of the λ_{j_b} -eigenvectors for A . The last quantity in this equation is just a shorthand notation for the previous one.

One can use these estimates to get inequalities between sums of eigenvalues of $A+B$ and related sums for A and B individually. Suppose that we can find a subspace V that belongs to the intersection of Schubert varieties $\mathcal{S}_{\mathcal{F}_{(A+B),K}} \cap \mathcal{S}_{\mathcal{F}_{A,I}^{\text{opp}}} \cap \mathcal{S}_{\mathcal{F}_{B,J}^{\text{opp}}}$ for some jump sequences I, J , and K . Then, as above in the case when U was a line, we can conclude that

$$(2.26) \quad \begin{aligned} \text{tr}_K(A+B) &= \sum_{b=1}^d \lambda_{k_b}(A+B) \leq \text{tr}(P_V(A+B)) = \text{tr}(P_V A) + \text{tr}(P_V B) \\ &\leq \sum_{c=1}^d \lambda_{n-j_c+1}(A) + \sum_{c=1}^d \lambda_{n-k_c+1}(B) = \text{tr}_{I^{\text{opp}}}(A) + \text{tr}_{J^{\text{opp}}}(B). \end{aligned}$$

It turns out that the inequalities (2.26), for all d , provide necessary and sufficient conditions to characterize the eigenvalues of $A+B$.

Let $\{\lambda_j\}$, $\{\mu_j\}$, and $\{\nu_j\}$, for $1 \leq j \leq n$, be three decreasing sequences of real numbers. Let J be a subset of the whole numbers from 1 to n . Suppose that the elements of J are listed in a vector, as in (2.1). We set

$$(2.27) \quad \lambda_J = \sum_{j \in J} \lambda_j = \sum_{a=1}^d \lambda_{j_a}.$$

We define μ_J and ν_J similarly. We let N stand for the full set of positive integers up to n ,

$$N = \{1, 2, 3, \dots, n - 1, n\}.$$

Theorem 2.1. *In order that a sequence $\{\nu_j\}$ be the eigenvalue sequence of the sum $A + B$ of two Hermitian $n \times n$ matrices A and B whose eigenvalue sequences are $\{\lambda_j\}$ and $\{\mu_j\}$, respectively, it is necessary and sufficient that*

$$\nu_N = \lambda_N + \mu_N; \text{ and}$$

$$\nu_K \leq \lambda_{I^{\text{opp}}} + \mu_{J^{\text{opp}}}$$

for all jump sequences I, J , and K of length d , $1 \leq d \leq n$, such that the the Littlewood-Richardson coefficient $c_{\alpha_J, \alpha_K}^{\alpha_I^{\text{opp}}}$ is non-zero.

The necessity of the conditions of Theorem 2.1 follows from the discussion above, specifically from the inequalities (2.26) and the intersection-theoretic interpretation of the LR coefficients described in §2.3. The sufficiency is a consequence of the Horn Conjecture (see references cited above). Theorem 2.1 is not exactly the Horn Conjecture, which was formulated in a rather technical way in terms of recursion on n , the size of the matrices. Rather, Theorem 2.1 might be thought of as the essential geometric solution to the eigenvalue question, while the specifics of Horn’s conjecture provides further information about the recursive nature of the triples α, β, γ of partitions such that the LR coefficients $c_{\alpha, \beta}^\gamma$ are non-zero.

One might wonder why the inequalities (2.26) should be a sufficient set of conditions to characterize the eigenvalues of $A + B$. There are at least two aspects to this question. First, why should linear inequalities among eigenvalues be enough for the characterization? Second, why these inequalities?

As to the first question, the best answer seems to lie in the general “yoga” of symplectic geometry and, in particular, of convexity properties of the moment map for Hamiltonian actions of groups ([GS], [Kir]). These considerations apply to the eigenvalues-of-the-sum problem, and they imply that the collection of possible eigenvalue sequences for $A + B$ should be a convex polyhedron in \mathbb{R}^n . Hence, it will be described by some collection of linear inequalities among the coordinates.

As to the second question, it is probably germane that the space of skew Hermitian matrices, which are just $\sqrt{-1}$ times Hermitian matrices, is the Lie algebra of the unitary group $U(n)$. A standard fact in Lie theory ([Ho1]) says that to any representation

$$\rho : U(n) \rightarrow U_Y,$$

where U_Y is the unitary group of a complex vector space Y endowed with a Hermitian inner product, there is an associated “infinitesimal” representation of the Lie algebra: $d\rho : \mathfrak{u}_n \rightarrow \mathfrak{u}_Y$. The mapping $d\rho$ will send skew Hermitian $n \times n$ matrices to skew Hermitian operators on Y . Since $d\rho$ is linear, it will preserve sums:

$$d\rho(\sqrt{-1}A + \sqrt{-1}B) = \sqrt{-1}(d\rho(A) + d\rho(B))$$

(where we have abused notation somewhat to write $d\rho(A) = (\frac{1}{\sqrt{-1}})d\rho(\sqrt{-1}A)$ for a Hermitian matrix A). Thus, the unitary representations of $U(n)$ attach a large family of other such sums to a sum of Hermitian operators.

Among the representations of $U(n)$, the natural actions on the exterior powers $\Lambda^d(\mathbb{C}^n)$ are especially significant examples. Given a Hermitian matrix A , its image $d\Lambda^d(A)$ will be a Hermitian matrix on $\Lambda^d(\mathbb{C}^n)$. Moreover, the eigenvalues of $d\Lambda^d(A)$ will be exactly the set of all d -fold sums of eigenvalues of A . Thus, inequalities

among eigenvalues of $d\Lambda^d(A)$ and $d\Lambda^d(B)$, and $d\Lambda^d(A+B)$, will translate directly into inequalities among sums of eigenvalues for A , B , and $A+B$. This suggests that it would be reasonable to look for inequalities of this kind.

Furthermore, the $\Lambda^d(\mathbb{C}^n)$ play a distinguished role in the representation theory of $U(n)$ (as we will see later in this paper). They often are called the *fundamental representations* of $U(n)$, because any other representation of $U(n)$ can be constructed recursively from them by applications of tensor product and (virtual) direct sum. This perhaps makes it reasonable that eigenvalue inequalities stemming from the $d\Lambda^d(A)$ should be particularly important in a result such as Theorem 2.1. At any rate, this result does involve the Littlewood-Richardson coefficients directly in the solution of a seductive problem in operator theory.

2.5. Extensions of abelian groups. A finite abelian group is canonically a product of groups of prime power order. It is standard that a finite abelian p -group can be expressed as a direct sum of cyclic subgroups $C_{p^{\alpha_i}}$ of order p^{α_i} . Furthermore, in any direct sum decomposition $A \simeq \prod_{i=1}^r C_{p^{\alpha_i}}$, the number of indices i for which α_i takes on a given value depends only on A , not on the specific decomposition. We will call the exponents involved in the cyclic decomposition of a finite abelian p -group the *exponents* of A . If we arrange the exponents of A in decreasing order, they will form a partition of α , where $\#(A) = p^\alpha$ is the order of A . Let us denote this partition by $P(A)$.

If C is a finite abelian p -group and $A \subset C$ is a subgroup, then we can form the quotient $B = C/A$. A natural question in this context is, what is the relationship between the partitions of A , B , and C ? It is not straightforward: clearly, the size of $P(C)$ is the sum of the sizes of $P(A)$ and $P(B)$, but the individual parts of $P(C)$ can vary substantially for given $P(A)$ and $P(B)$. For example, if $A \cong \mathbb{Z}/p\mathbb{Z} \cong B$, then C can be isomorphic either to $(\mathbb{Z}/p\mathbb{Z})^2$ or to $\mathbb{Z}/p^2\mathbb{Z}$. It turns out that, if $P(A) = D$ and $P(B) = E$, then for a third partition F , there is an extension C of A by B such that $P(C) = F$ if and only if the LR coefficient $c_{D,E}^F$ is non-zero! (We note that the parameters describing irreducible representations of $GL_n(\mathbb{C})$ can effectively be taken to be partitions; see the discussion just below in §3.) An algebraic approach to this question was taken by P. Hall. This is discussed in [Ma]. See also [Fu2, §2].

3. REPRESENTATIONS OF GL_n

In this section, we shall review some basic facts in the representation theory of $GL_n = GL_n(\mathbb{C})$ ([GW], [Hum], [W2]).

Let V be a finite-dimensional complex vector space and let $\rho : GL_n \rightarrow GL(V)$ be a *rational* representation of GL_n . This means that ρ is a group homomorphism, and for any vector \mathbf{v} in V and any linear functional λ on V , the function on G

$$g \rightarrow \lambda(\rho(g)(\mathbf{v}))$$

is regular; that is, it is a polynomial in the entries of g and $1/(\det g)$. Let B_n be the standard Borel subgroup of upper triangular matrices in GL_n . The Lie-Kolchin Theorem ([Hum]) says that $\rho(B_n)$ has an eigenvector v_0 , that is,

$$\rho(b)(v_0) = \psi(b)v_0 \quad (b \in B_n),$$

where $\psi : B_n \rightarrow \mathbb{C}^\times$ is a character of B_n . The Borel subgroup B_n can be written as $B_n = A_n U_n$ where A_n is the diagonal torus of GL_n and U_n is the maximal unipotent subgroup consisting of all the upper triangular matrices with 1's on the

diagonal. In fact, U_n is the commutator subgroup of B_n , so that $\psi(u) = 1$ for all $u \in U_n$. Thus

$$\rho(u)(v_0) = v_0 \quad (u \in U_n)$$

and ψ is determined by its restriction to A_n . Now the rational characters of A_n can be parametrized by \mathbb{Z}^n : for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, let $\psi_n^\alpha : A_n \rightarrow \mathbb{C}^\times$ be defined by

$$(3.1) \quad \psi_n^\alpha[\text{diag}(a_1, \dots, a_n)] = a_1^{\alpha_1} \cdots a_n^{\alpha_n}.$$

Here $\text{diag}(a_1, \dots, a_n)$ is the $n \times n$ diagonal matrix such that its diagonal entries are a_1, \dots, a_n . Let

$$(3.2) \quad \Lambda_n^+ = \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n : \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$$

and

$$(3.3) \quad \widehat{A}_n^+ = \{\psi_n^\alpha : \alpha \in \Lambda_n^+\}.$$

We call the characters in \widehat{A}_n^+ the *dominant weights* of GL_n . Then it is well known that the character ψ defined by the B_n -eigenvector v_0 belongs to \widehat{A}_n^+ ,

$$\psi(a) = \psi_n^\lambda(a) \quad (a \in A_n)$$

for some $\lambda \in \Lambda_n^+$, and we call v_0 a *highest weight vector* of weight ψ_n^λ . If ρ is irreducible, then the vector v_0 is unique up to scalar multiples. In this case, we call the character ψ_n^λ of A_n the *highest weight* of ρ , and it determines the representation ρ uniquely. In view of this, we shall denote ρ by ρ_n^λ . For example, if $V = \mathbb{C}$, $m \in \mathbb{Z}$, and

$$\rho(g)(v) = (\det g)^m v \quad g \in \text{GL}_n, v \in \mathbb{C},$$

then $\rho = \rho_n^{m\mathbf{1}_n}$ where

$$(3.4) \quad \mathbf{1}_n = (\overbrace{1, 1, \dots, 1}^n) \quad \text{and} \quad m\mathbf{1}_n = (\overbrace{m, m, \dots, m}^n).$$

The set $\{\rho_n^\lambda : \lambda \in \Lambda_n^+\}$ exhausts all the irreducible rational representations of GL_n .

If $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n^+$ is such that $\lambda_n \geq 0$, then we call ρ_n^λ a *polynomial representation* of GL_n . Every irreducible rational representation ρ_n^α of GL_n is isomorphic to the tensor product of a polynomial representation and a power of determinant, that is,

$$\rho_n^\alpha \cong \rho_n^\lambda \otimes \rho_n^{m\mathbf{1}_n}$$

for some polynomial representation ρ_n^λ and integer $m \in \mathbb{Z}$. For example, we can take $\lambda = (\alpha_1 - \alpha_n, \alpha_2 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n, 0)$ and $m = \alpha_n$.

In general, a rational representation $\rho : \text{GL}_n \rightarrow \text{GL}(V)$ can be decomposed as a direct sum of irreducible representations:

$$(3.5) \quad V = \bigoplus_{\alpha \in \Lambda_n^+} (\text{Hom}_{\text{GL}_n}(\rho_n^\alpha, V) \otimes \rho_n^\alpha).$$

Here we abuse notation and denote the representation space of ρ_n^α also by ρ_n^α , and $\text{Hom}_{\text{GL}_n}(\rho_n^\alpha, V)$ is the space of all GL_n intertwining maps $T : \rho_n^\alpha \rightarrow V$, that is, $T(\rho_n^\alpha(g)(v)) = \rho(g)T(v)$ for all $g \in \text{GL}_n$ and $v \in \rho_n^\alpha$. The group GL_n acts on $\text{Hom}_{\text{GL}_n}(\rho_n^\alpha, V)$ trivially. The *multiplicity* of ρ_n^α in V is defined as

$$m(V, \rho_n^\alpha) = \dim \text{Hom}_{\text{GL}_n}(\rho_n^\alpha, V).$$

Equation (3.5) can also be written as

$$V = \bigoplus_{\alpha \in \Lambda_n^+} m(V, \rho_n^\alpha) \rho_n^\alpha.$$

The multiplicities $m(V, \rho_n^\alpha)$ are closely related to the highest weight vectors in V of weight ψ_n^α . Let

$$V^{U_n} = \{v \in V : \rho(u)(v) = v \ \forall u \in U_n\}.$$

Then V^{U_n} is stable under the action by A_n , so it can be decomposed as

$$V^{U_n} = \bigoplus_{\alpha \in \Lambda_n^+} V_\alpha^{U_n},$$

where for each $\alpha \in \Lambda_n^+$,

$$V_\alpha^{U_n} = \{v \in V^{U_n} : \rho(a)(v) = \psi_n^\alpha(a)v \ \forall a \in A_n\}$$

is the ψ_n^α -eigenspace of A_n . Each non-zero vector v in $V_\alpha^{U_n}$ is a highest weight vector of weight ψ_n^α and it defines an intertwining map $T_v : \rho_n^\alpha \rightarrow V$ as follows. Fix a highest weight vector w_α in ρ_n^α . Then there is a unique GL_n intertwining map $T_v : \rho_n^\alpha \rightarrow V$ with the property that $T_v(w_\alpha) = v$. In this way, we obtain a linear map $V_\alpha^{U_n} \rightarrow \text{Hom}_{GL_n}(\rho_n^\alpha, V)$ which sends v to T_v . This map is a vector space isomorphism, so we have

$$(3.6) \quad m(V, \rho_n^\alpha) = \dim V_\alpha^{U_n}.$$

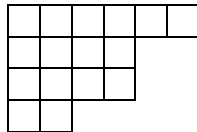
4. ANATOMY OF THE LR RULE

We will now give a statement of the Littlewood-Richardson Rule. This takes a little preparation.

A *Young diagram* D is an array of square boxes arranged in left-justified horizontal rows, with each row no longer than the one above it ([Fu1]). If D has at most m rows, then we shall denote it by

$$D = (\lambda_1, \dots, \lambda_m),$$

where for each i , λ_i is the number of boxes in the i th row of D . For example, the following is the Young diagram $(6, 4, 4, 2)$.



We shall denote the number of rows in D by $r(D)$ and the total number of boxes in D by $|D|$. If $|D| = n$, then the sequence $(\lambda_1, \dots, \lambda_m)$ is also called a *partition* of n .

If one Young diagram D sits inside another Young diagram F , then we write $D \subset F$. In this case, by removing all boxes belonging to D , we obtain the *skew diagram* $F - D$. If we put a positive number in each box of $F - D$, then it becomes a *skew tableau* and we say that the *shape* of this skew tableau is $F - D$. If the entries of this skew tableau are taken from $\{1, 2, \dots, m\}$ and μ_j of them are j for $1 \leq j \leq m$, then we say the *content* of this skew tableau is $E = (\mu_1, \dots, \mu_m)$. If T is a skew tableau, then the *word* of T is the sequence $w(T)$ of positive integers

obtained by reading the entries of T from top to bottom and right to left in each row. For example,

$$(4.1) \quad T = \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 & 2 \\ \hline & & 2 & 3 & & \\ \hline & 1 & 3 & 4 & & \\ \hline 1 & 2 & & & & \\ \hline \end{array}$$

is a skew tableau of shape $F - D$ and content E , where $D = (3, 2, 1)$, $F = (6, 4, 4, 2)$, and $E = (4, 3, 2, 1)$, and its word is given by

$$w(T) = (2, 1, 1, 3, 2, 4, 3, 1, 2, 1).$$

A *Littlewood-Richardson* (LR) tableau is a skew tableau T with the following properties:

- (i) It is *semistandard*; that is, the numbers in each row of T weakly increase from left-to-right, and the numbers in each column of T strictly increase from top-to-bottom.
- (ii) It satisfies the *Yamanouchi word condition* (YWC); that is, for each positive integer j , starting from the first entry of $w(T)$ to any place in $w(T)$, there are at least as many j 's as $(j + 1)$'s.

Note that for semistandard tableaux T , (ii) is equivalent to:

- (ii)' The number of j 's in the first a rows of T is at least as large as the number of $(j + 1)$'s in the first $a + 1$ rows, for $j, a \geq 1$.

This is because, for a semistandard tableau, the part of the word coming from a given row has entries that are weakly decreasing. This means that all the $(j + 1)$'s in row $(a + 1)$ appear before any of the j 's in that row. Thus, the j 's that appear before the $(j + 1)$'s of row $(a + 1)$ must be supplied by the first a rows, and the YWC requires the number of these j 's to be larger than the number of the $(j + 1)$'s in the first $(a + 1)$ rows.

Here are some examples. The skew tableau T given in (4.1) is semistandard, but it does not satisfy the YWC, so it is not an LR tableau. On the other hand, one can check that the following skew tableau T' is an LR tableau.

$$T' = \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline & & 1 & 2 & & \\ \hline & 2 & 2 & 3 & & \\ \hline 3 & 4 & & & & \\ \hline \end{array}$$

For Young diagrams D, E , and F , the *LR coefficient* $c_{D,E}^F$ is defined as

$$(4.2) \quad c_{D,E}^F = \text{the number of LR tableaux of shape } F - D \text{ and content } E.$$

Young diagrams with at most n rows can be used to label the polynomial representations of GL_n . Let D be such a Young diagram. Then it can be identified with an n -tuple of integers $(\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$, so it corresponds to the dominant weight $\psi_n^{(\lambda_1, \dots, \lambda_n)}$ and the polynomial representation $\rho_n^{(\lambda_1, \dots, \lambda_n)}$ of GL_n . We shall write $\psi_n^{(\lambda_1, \dots, \lambda_n)}$ and $\rho_n^{(\lambda_1, \dots, \lambda_n)}$ as ψ_n^D and ρ_n^D , respectively.

We now consider two polynomial representations ρ_n^D and ρ_n^E of GL_n and form their tensor product $\rho_n^D \otimes \rho_n^E$. The Littlewood-Richardson Rule gives a description of the multiplicities of this tensor product.

THE LITTLEWOOD-RICHARDSON RULE

The multiplicity of ρ_n^F in the tensor product $\rho_n^D \otimes \rho_n^E$ is given by the LR coefficient $c_{D,E}^F$.

5. THE PIERI RULE AND THE LR RULE

The case of tensoring a general irreducible representation of GL_n with a representation corresponding to a Young diagram with one row (equivalently, with a symmetric power of the standard representation), is easy to describe. It is well known classically as the *Pieri Rule*.

Let D be a Young diagram, and consider the representation ρ_n^D of GL_n corresponding to D . Let S^a denote the a th symmetric power of the standard action of GL_n on \mathbb{C}^n . Then $S^a \simeq \rho_n^{R_a}$, where R_a is the single row containing a boxes. The Pieri Rule first says that $\rho_n^D \otimes S^a$ is multiplicity free: any irreducible representation of GL_n appears in $\rho_n^D \otimes S^a$ at most one time. Second, it says that the representations that do appear in $\rho_n^D \otimes S^a$ are the ρ_n^E , where E is a diagram such that

- i) $D \subset E$; and
- ii) $E - D$ is a skew-row containing a boxes.

The term “skew-row” means that any column of E is at most one box longer than the corresponding column of D , or equivalently, that each column of E contains at most one box that is not in D . This includes the possibility that E has a column of length one where D had no column at all, i.e., the first row of E may be strictly longer than the first row of D . In Figures 5.1 and 5.3, the numbered boxes form a skew row, but in Figures 5.2 and 5.4 they do not.

From these conditions, we can see what the result will be if we tensor with several symmetric powers S^{a_j} in succession. We consider the multiple tensor product

$$(5.1) \quad \Omega = \rho_n^D \otimes \left(\bigotimes_{j=1}^k S^{a_j} \right).$$

Each constituent of Ω will be described by a nested sequence $D = D_0 \subset D_1 \subset D_2 \subset \dots \subset D_k$ of Young diagrams, with each difference $D_j - D_{j-1}$ being a skew row.

We can also describe such a succession by means of skew tableaux. Fill the boxes of $D_1 - D_0$ with 1's. Fill the boxes of $D_2 - D_1$ with 2's. And so on. The result will be that the skew diagram $D_k - D_0$ is filled with the numbers 1 through k : we have constructed a skew tableau that describes a constituent of the multiple tensor product. Furthermore, the condition that $D_j \supset D_{j-1}$ implies that the entries of the tableau are weakly increasing along any row. Also, the condition that $D_j - D_{j-1}$ is a skew row implies that the entries of the tableau are strictly increasing down any column. In other words, the skew tableaux that describe the successive tensor products with symmetric powers are semistandard. It is also not hard to convince oneself that, given a semistandard skew tableau living in the skew diagram $D_k - D_0$, there is a succession of nested subdiagrams D_j such that $D_j - D_{j-1}$ is a skew row; namely D_j is the diagram whose boxes contain numbers $i \leq j$. The condition of weakly increasing indices along rows implies the nestedness of the D_j , and the condition of strictly increasing indices down columns implies that $D_j - D_{j-1}$ is a skew row. Thus, on the one hand, nested sequences of diagrams satisfying condition

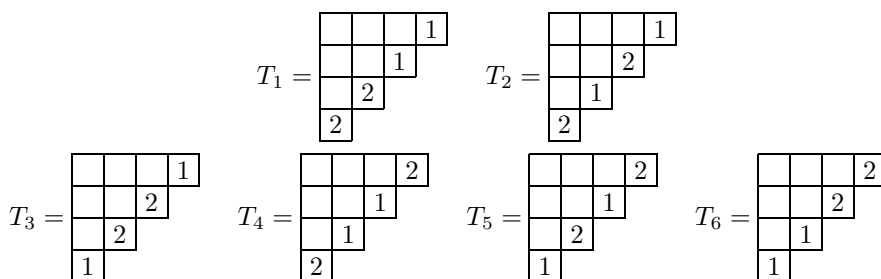


FIGURE 5.1. Semistandard tableaux of shape $D_2 - D_0$ and content $(2, 2)$.

ii) at each step and, on the other hand, semistandard skew tableaux are two ways of describing the same set—the set of constituents of the multiple tensor product Ω of formula (5.1).

Here is an example. Let $n \geq 4$, $D = D_0 = (3, 2, 1)$, and $D_2 = (4, 3, 2, 1)$. Then $D_2 - D_0$ is a skew row, consisting of the bottom box in each column of D_2 . Consider the tensor product $\rho_n^D \otimes S^2 \otimes S^2$. There are six semistandard tableaux of shape $D_2 - D_0$ and content $(2, 2)$; see Figure 5.1. It follows that the multiplicity of $\rho_n^{D_2}$ in the tensor product $\rho_n^D \otimes S^2 \otimes S^2$ is 6.

An analogous situation holds if we look at tensor products with exterior powers Λ^b of \mathbb{C}^n . A tensor product $\rho_n^D \otimes \Lambda^b$ is always multiplicity free. Its constituents consist of representations ρ_n^E such that

- i) $D \subset E$; and
- ii) $E - D$ is a skew column containing b boxes.

The term “skew column” means that each row of E contains at most one box not contained in D , or equivalently, that each row of E is at most one box longer than the same row of D . This includes the possibility that E has a row of length one where D had no row at all; that is, the first column of E may be strictly longer than the first column of D .

From these conditions, we can see what the result will be if we tensor with several exterior powers Λ^{b_j} in succession. Consider the multiple tensor product

$$(5.2) \quad \Psi = \rho_n^D \otimes \left(\bigotimes_{j=1}^r \Lambda^{b_j} \right).$$

Each constituent of Ψ will be described by a nested sequence $D = D_0 \subset D'_1 \subset D'_2 \subset \dots \subset D'_r$ of Young diagrams, with each difference $D'_j - D'_{j-1}$ being a skew column.

We could identify such a nested sequence with a tableau by filling $D'_j - D'_{j-1}$ with the number j . Just as in the case of tensoring with symmetric powers, this tableau would allow us to reconstruct the sequence D'_i , and so would uniquely label the constituents of the multiple tensor product.

However, this labeling will tend to produce skew diagrams that are not semistandard. We want to consider an alternative labeling scheme for tableaux to record the constituents. In the skew column of $D'_j - D'_{j-1}$, of length b_j , we will put the numbers 1 through b_j consecutively down the skew column. We will call this the *standard filling* of a skew column.

			1
	2	1	
2	3		

FIGURE 5.2. A non-peelable tableau satisfying YWC.

		1
2	1	
2		

FIGURE 5.3. A peelable, but not standard-peelable, tableau.

Unfortunately, this labeling of the skew columns of a constituent of a multiple tensor product does not uniquely determine the constituents of $\rho_n^{D_0} \otimes \left(\bigotimes_{j=1}^r \Lambda^{b_j} \right)$; it is easy to produce examples where different sequences of nested diagrams produce the same tableau. For example, the tableau T_1 in Figure 5.1 can be produced by four standard fillings. However, standard filling does produce tableaux that satisfy the YWC; clearly with this scheme, for every box labeled j , there is a box labeled $j - 1$ in a higher row.

In fact, giving each skew column the standard filling produces a tableau that satisfies a stronger condition than the YWC: it is *peelable*. If we have a skew tableau in a skew diagram $E - D$, then we say we can *peel* a skew column off the tableau, if we can find a diagram E_1 , with $D \subset E_1 \subset E$, such that

- i) $E - E_1$ is a skew column, and
- ii) $E - E_1$ has the standard filling.

We say that the tableau $E - D$ is *peelable* if we can find a sequence of peelings that exhausts $E - D$. That is, we can find a sequence $E = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_r = D$ such that each difference $C_j = E_j - E_{j+1}$ is a skew column with the standard labeling.

Among all skew columns that can be peeled off a diagram, there will be a longest possible length. Among all longest peelable skew columns, there will be one with a 1 in the highest possible row, a 2 in the highest possible row below that, a 3 in the highest possible row below that, and so forth. We will call this the *standard-peelable column*. We will say that a tableau is *standard peelable* if it allows a peeling by standard-peelable columns.

The following is evident:

- Lemma 5.1.** (a) *The tableaux obtained by filling the successive skew columns of a constituent of Ψ with the standard filling will produce a peelable tableau.*
 (b) *A peelable tableau satisfies YWC.*

We can see by example that YWC does not imply peelability, and that a peelable tableau may not be standard peelable. See Figures 5.2 and 5.3. It is also easy to see that a tableau satisfying YWC or that is peelable or even standard peelable, need not be semistandard. Indeed, the Figures 5.2 and 5.3 are not semistandard, and Figure 5.4 is a standard-peelable tableau that is also not semistandard.

	1
	1
2	
2	

FIGURE 5.4. A standard-peelable, but not semistandard, tableau.

However, in [HTW3, §2.3.3], the following result is proved:

Proposition 5.2. *A semistandard tableau satisfying YWC is standard peelable. Moreover, the standard-peelable column at any stage is (weakly) longer than the standard-peelable column of the next stage.*

It is worth remarking also that, although a peelable tableau might not be uniquely peelable, a tableau can have at most one standard peeling, by virtue of the definition of standard peeling.

We have been discussing how to decompose the tensor products Ω of formula (5.1) or Ψ of formula (5.2) by starting with the representation ρ_n^D and tensoring successively with symmetric (resp., exterior) powers. However, we may alternatively regard Ω as the tensor product of ρ_n^D with the single representation $\bigotimes_{j=1}^k S^{a_j}$, gotten by tensoring all the symmetric powers together. This representation is highly reducible; its irreducible components are found by applying the principles above to the case of $D = \emptyset$, the empty diagram. Thus, the constituents are described by all semistandard tableaux containing a_j boxes filled with j . Suppose that the row lengths a_j are weakly decreasing, so that they are the lengths of the rows of a Young diagram E . Then there is exactly one such tableau that fills the diagram E . This is the tableau in which all boxes in the j th row of E are filled with j .

We can make a qualitative statement about the other constituents of $\bigotimes_{j=1}^k S^{a_j}$. Recall that the set of Young diagrams has a partial ordering. It can be defined in terms of moving boxes or in terms of row lengths. In terms of row lengths, we say that given two diagrams F and G , then $F \geq G$ provided first ([Fu1]), that F and G have the same number of boxes and second, the sum of the lengths of the first r rows of F is at least as large as the sum of the lengths of the first r rows of G , for each positive integer r .

Proposition 5.3. (a) *If the a_j are the lengths of the rows of the diagram E , then the tensor product $\bigotimes_{j=1}^k S^{a_j}$ contains the representation ρ_n^E with multiplicity one.*

(b) *If ρ_n^G is a constituent of $\bigotimes_{j=1}^k S^{a_j}$, then $G \geq E$.*

Proof. Since each time we tensor with a symmetric power, we add a skew row to a diagram and we see that, if we start with the empty diagram, then after tensoring with s factors S^{a_j} , we can only obtain diagrams with at most s rows. The number of boxes in these diagrams will be the sum of the a_j for $1 \leq j \leq s$, that is, it is equal to the number of boxes in the first s rows of E . Hence, the number of boxes in the first s rows of any tableau produced by this process is at least equal to the number of boxes in the first s rows of E . Since this is true for each s , if we take into

account our description of the diagram ordering, statement b) of the proposition follows directly. \square

On the other hand, if we make the analogous construction with columns, we get a quite different set of constituents.

Proposition 5.4. (a) *If the b_j are the lengths of the columns of the diagram E , then the tensor product $\bigotimes_{j=1}^r \Lambda^{b_j}$ contains the representation ρ_n^E with multiplicity one.*
 (b) *If ρ_n^G is a constituent of $\bigotimes_{j=1}^r \Lambda^{b_j}$, then $E \geq G$.*

Proof. The diagram G is built up out of skew columns of lengths equal to the columns of E . The skew column based on the column of E of length b_j can have at most $\min(s, b_j)$ boxes in the first s rows of G . Thus, the number of boxes in the first s rows of G will not exceed the number of boxes in the first s rows of E , which means that $E \geq G$, as claimed. The only way to get exactly E by this process is for each column of E to fill consecutive rows of the diagram, starting with the first row. Thus, the multiplicity of E is one. \square

Corollary 5.5. *If the diagram E has rows of lengths a_j and columns of lengths b_ℓ , then the multiple tensor products $\bigotimes_{j=1}^k S^{a_j}$ and $\bigotimes_{\ell=1}^r \Lambda^{b_\ell}$ have exactly one irreducible constituent in common, namely one copy of ρ_n^E .*

Remarks. a) Corollary 5.5 is closely related to the reasons for the success of the construction of representations of the symmetric group by Young symmetrizers ([W2], [Fu1]), and it is linked by Schur duality to the use of Young symmetrizers to construct the representations of GL_n .

b) Propositions 5.3 and 5.4, as well as Corollary 5.5 should perhaps be taken as part of the folklore of the subject. We have not conducted a search to find them in the literature.

c) Proposition 5.3 is also related to the decomposition (7.13), which exhibits multiplicities in $\bigotimes_{j=1}^k S^{a_j}$ as equal to multiplicities of weight spaces for a maximal torus in irreducible representations. These multiplicities are known as Kostka numbers ([Fu1]). This interpretation translates Proposition 5.3 into well-known facts about Kostka numbers (i.e., the unitriangularity of the Kostka matrix with respect to the dominance order on Young diagrams) ([Fu1]).

The above considerations can help us to accept that the combination of semistandardness and the YWC might be the appropriate conditions to put on tableaux in order to count constituents of a tensor product. Let D and E be two Young diagrams. Let a_j be the lengths of the rows of E , and let b_ℓ be the lengths of the columns of E . Consider the tensor products Ω of formula (5.1) and Ψ of formula (5.2). By Propositions 5.3 and 5.4, both of these tensor products contain the tensor product $\rho_n^D \otimes \rho_n^E$. Moreover, Ω is a sum of tensor products $\rho_n^D \otimes \rho_n^G$ over diagrams $G \leq E$, and Ψ is a sum of tensor products $\rho_n^D \otimes \rho_n^H$ over diagrams $H \geq E$. Except for the unique summand $\rho_n^D \otimes \rho_n^E$, none of the summands constituting Ω is the same as any of the summands constituting Ψ .

We have seen how to label the constituents of Ω with semistandard skew tableaux. We have also seen how to label the constituents of Ψ by peelable skew tableaux. This latter labeling is non-unique, but for the subset of standard-peelable tableaux,

there is only one possible standard peeling.¹ If we would like to believe that these labelings are not purely formal devices, but capture something significant in the structure of these representations, we might hope that the skew tableaux which appear in both labelings are exactly the skew tableaux that describe the unique common constituent $\rho_n^D \otimes \rho_n^E$ of Ω and Ψ . According to Proposition 5.2, this is exactly the assertion of the Littlewood-Richardson Rule.

6. THE TWO-ROWED CASE

The original paper of Littlewood and Richardson ([LR]) established the LR Rule in the case of tensor products $\rho_n^D \otimes \rho_n^E$, where E has two rows. In this section, we will revisit their argument. It will be shown in §7 that the two-rowed case, applied in a slightly more general context, in fact implies the general case.

As noted in §5, we know how to tensor with one-rowed representations: the Pieri Rule tells us how to find the constituents. By iterating this process, we can find the tensor products $\rho_n^D \otimes \rho_n^E$ for any diagram E , but hidden in a sum that includes many other tensor products. This gives us an approach to determining $\rho_n^D \otimes \rho_n^E$: try to isolate it in the multiple tensor product Ω . This is the strategy of Littlewood and Richardson for two-rowed diagrams E .

Let $\rho_n^{(a,b)}$ denote the representation attached to the two-rowed Young diagram with rows of lengths a and $b \leq a$. The basic observation is that

(6.1)

$$S^a \otimes S^b \simeq \bigoplus_{j=0}^b \rho_n^{(a+b-j,j)} \simeq \rho_n^{(a,b)} \oplus \left(\bigoplus_{j=0}^{b-1} \rho_n^{(a+b-j,j)} \right) \simeq \rho_n^{(a,b)} \oplus (S^{a+1} \otimes S^{b-1}).$$

This says that to find the constituents of $\rho_n^D \otimes \rho_n^{(a,b)}$, we can look at the constituents of $\rho_n^D \otimes S^a \otimes S^b$ and try to figure out which of them could not come from $\rho_n^D \otimes S^{a+1} \otimes S^{b-1}$.

We know that the constituents of $\rho_n^D \otimes S^a \otimes S^b$ are labeled by a sequence $D \subset E \subset F$ of diagrams, such that $E - D$ and $F - E$ are skew rows with a boxes and b boxes, respectively. Consider first the case when $F - D$ is a skew row and a skew column at the same time. This means that the $a + b$ boxes of $F - D$ are distributed in $a + b$ different rows and $a + b$ different columns. This is the situation in Figure 5.1.

A constituent of $\rho_n^D \otimes S^a \otimes S^b$ isomorphic to ρ_n^F is specified by distributing a 1's and b 2's in the $a + b$ boxes of $F - D$. This distribution of 1's and 2's must make $F - E$ into a semistandard skew tableau. Since we are assuming that $F - E$ is both a skew row and a skew column, the semistandard conditions are vacuous, and *any* distribution of a 1's and b 2's among the $a + b$ boxes is allowable. Such a distribution is determined by deciding where to put the 1's, so there are in all

¹Another aspect of non-uniqueness affects both labelings: the tensor products $\bigotimes_{j=1}^k S^{a_j}$ and $\bigotimes_{\ell=1}^r \Lambda^{b_\ell}$ do not depend on the order of the factors. However, the labelings that will result from the procedures defined above *do* depend on the order of the factors. For example, a_j counts the number of j s that appear in the skew tableau of Ω . If we change the order of the factors in $\bigotimes_{j=1}^k S^{a_j}$, we will change the content of the resulting tableaux. In particular, we have no chance of getting any tableaux satisfying the YWC unless the a_j are arranged in decreasing order—that is, unless they form the row lengths of a Young diagram. Similarly, in order for a constituent of $\bigotimes_{\ell=1}^r \Lambda^{b_\ell}$ to correspond to the standard peeling of a peelable semistandard tableau, the b_ℓ must be arranged in increasing order.

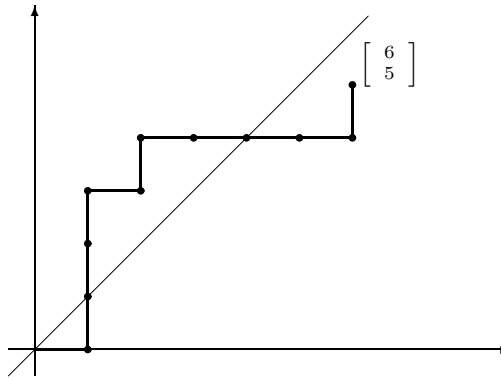


FIGURE 6.1. The path corresponding to the sequence (1, 2, 2, 2, 1, 2, 1, 1, 1, 1, 2)

$\binom{a+b}{a}$ possible tableaux. Similarly, if we look at components of $\rho_n^D \otimes S^{a+1} \otimes S^{b-1}$ isomorphic to ρ_n^F , we see that there are $\binom{(a+1)+(b-1)}{a+1} = \binom{a+b}{a+1}$. In order for the LR formula to hold in this case, we should therefore have that the number of LR tableaux among all the possible tableau, which is to say, the number of tableaux satisfying YWC should be $\binom{a+b}{a} - \binom{a+b}{a+1}$.

To see that this is true, it is helpful to think of the binomial coefficients in terms of paths in the plane. Consider paths in $(\mathbb{Z}^+)^2$ that start at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and at each step, move either one space to the right (add $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$) or one space up (add $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$). We will call these *increasing paths*. Then the number of increasing paths leading from the origin to $\begin{bmatrix} a \\ b \end{bmatrix}$ is just the binomial coefficient $\binom{a+b}{b}$. See Figure 6.1 for an example of an increasing path.

The YWC for the tableaux we are dealing with amounts to the requirement that if we list the contents of the boxes starting from the top, then at any stage in the list, we should have written at least as many 1's as 2's. If we map tableaux to paths by letting a 1 correspond to moving to the right, and letting a 2 correspond to moving up, then the YWC translates into the condition that our path should never go above the diagonal line $y = x$ in the plane. Thus, the following lemma tells us that we have the right number of LR tableaux.

Lemma 6.1. *The increasing paths from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} a \\ b \end{bmatrix}$ that go above the diagonal (that is, that pass through a point $\begin{bmatrix} m \\ n \end{bmatrix}$ with $n > m$), are in one-to-one correspondence with the increasing paths from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$.*

Remarks.

- a) The argument of Littlewood and Richardson does not use the imagery of paths, but it is formally equivalent to this one.
- b) The condition that a path not go above the diagonal is involved in the definition of Catalan numbers ([Sta]). Catalan numbers are concerned with the case $a = b$. An argument like the one below for this case can be found in the combinatorics literature. See for example [FH]. Despite some inquiries,

we did not find a reference for the case of general (a, b) . However, the argument is no different in the general case than when $a = b$.

- c) In this lemma, one can see the beginning of Littelmann's path model ([Lim]) for representations.

Proof. The main idea of the proof is illustrated in Figure 6.2. Let \mathcal{P} be an increasing path from the origin to $\begin{bmatrix} a \\ b \end{bmatrix}$ that rises above the main diagonal. We replace \mathcal{P} as in Figure 6.1 with a path \mathcal{P}' going to $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$ by strategically altering a single move of \mathcal{P} .

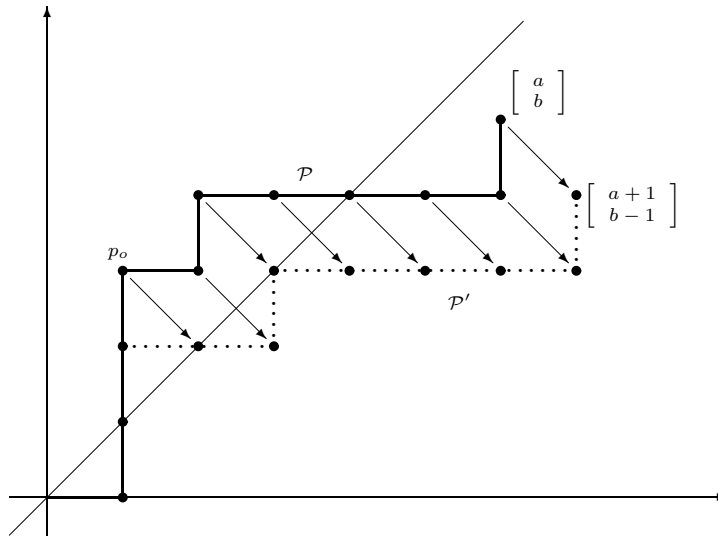
We want to keep track of the height of points above the diagonal. Thus, for a point $p = \begin{bmatrix} x \\ y \end{bmatrix}$ of the plane, let $h(p) = y - x$ be the vertical distance from p to the diagonal line.

We look at the first point $p_o = \begin{bmatrix} n \\ m \end{bmatrix}$ at which the path \mathcal{P} rises to the greatest height above the diagonal. That is, $h(p_o)$ is the maximum value of h among all points on the path, and n is the smallest first coordinate at which this maximum is attained. Notice that the origin is the first point on \mathcal{P} for which $h \geq 0$, so that if p_o is not the origin, then $h(p_o) > 0$; that is, p_o lies strictly above the diagonal.

Since p_o is the first point where h is maximal on \mathcal{P} , the path \mathcal{P} must have moved up to get to p_o ; that is, the point on \mathcal{P} just before p_o must be $\begin{bmatrix} m-1 \\ n \end{bmatrix}$. Also, since $h(p_o)$ is maximal on \mathcal{P} , the next move of \mathcal{P} must be to the right, that is, the next point after p_o on \mathcal{P} must be $\begin{bmatrix} m+1 \\ n \end{bmatrix}$.

We modify \mathcal{P} to get a different path \mathcal{P}' , as follows. Instead of moving up at $\begin{bmatrix} m-1 \\ n \end{bmatrix}$, \mathcal{P}' moves to the right. Thereafter, all the moves of \mathcal{P}' are the same as for \mathcal{P} . Thus, \mathcal{P}' has all the same moves as \mathcal{P} , except the $(m+n)$ -th move is right instead of up. In terms of points on the paths, the points of \mathcal{P} and \mathcal{P}' agree up to $\begin{bmatrix} m-1 \\ n \end{bmatrix}$, and thereafter, each point of \mathcal{P}' is obtained from the corresponding point of \mathcal{P} by shifting by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$: one step right and one step down. It is therefore clear that the endpoint of \mathcal{P}' will be $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$. Again, we refer to Figure 6.2.

We claim that this procedure produces a bijection between increasing paths from the origin to $\begin{bmatrix} a \\ b \end{bmatrix}$ that rise above the diagonal and all increasing paths to $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$. To see that this is true, consider what the path \mathcal{P} does after reaching $\begin{bmatrix} m \\ n \end{bmatrix}$. Since \mathcal{P} never moves above the diagonal containing $\begin{bmatrix} m \\ n \end{bmatrix}$, we see that after arriving at $\begin{bmatrix} m \\ n \end{bmatrix}$, the number of times that \mathcal{P} moves up never exceeds the number of times \mathcal{P} moves right. Since the moves of \mathcal{P}' from $\begin{bmatrix} m+1 \\ n-1 \end{bmatrix}$ are the same as the moves of \mathcal{P} from $\begin{bmatrix} m \\ n \end{bmatrix}$, it follows that, after leaving $\begin{bmatrix} m+1 \\ n-1 \end{bmatrix}$, \mathcal{P}' never rises above the diagonal containing $\begin{bmatrix} m+1 \\ n-1 \end{bmatrix}$. Since $\begin{bmatrix} m \\ n \end{bmatrix}$ is the first point on the highest diagonal reached by \mathcal{P} , it follows that $\begin{bmatrix} m \\ n-1 \end{bmatrix}$ is on the highest diagonal reached by \mathcal{P}' , and that it is the *last* point of \mathcal{P}' on this diagonal.

FIGURE 6.2. An illustration of the paths \mathcal{P} and \mathcal{P}'

This tells us how to reconstruct \mathcal{P} from \mathcal{P}' ; we look for the last point $\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix} \right]$ on the highest diagonal reached by \mathcal{P}' . The move from this point must be to the right, since moving up would take \mathcal{P}' to a higher diagonal. We replace the move to the right from $\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix} \right]$ with a move up, and leave all the other moves the same. This will reconstitute \mathcal{P} from \mathcal{P}' .

Any increasing path \mathcal{P}' to $\left[\begin{smallmatrix} a+1 \\ b-1 \end{smallmatrix} \right]$ from the origin always reaches the main diagonal (since it starts at the origin, which is on the main diagonal). Hence, the path \mathcal{P} constructed in the previous paragraph from \mathcal{P}' , will rise above the main diagonal. Also, it will clearly end at $\left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]$. Thus, we can construct an inverse to the map $\mathcal{P} \rightarrow \mathcal{P}'$, so the claim is established. The lemma follows directly from the claim. \square

Here is an example to illustrate the bijection described in Lemma 6.1. Again we consider the tensor product $\rho_n^D \times S^2 \otimes S^2$ where $D = (3, 2, 1)$. For $F = (4, 3, 2, 1)$, there are six semistandard tableaux T_1, \dots, T_6 of shape $F - D$ and content $(2, 2)$, as shown in Figure 5.1. Each of these tableaux can be identified with an increasing path from the origin to the point $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right]$. Among these paths, those which correspond to T_3, T_4, T_5 , and T_6 rise above the origin, and we alter them according to the procedure described in Lemma 6.1. The new paths now end at $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right]$, so they correspond to semistandard tableaux of shape $F - D$ with content $(3, 1)$. The explicit correspondence on tableaux in this case is given in Figure 6.3. In particular, T'_3, T'_4, T'_5, T'_6 give all the semistandard tableaux of shape $F - D$ and content $(3, 1)$.

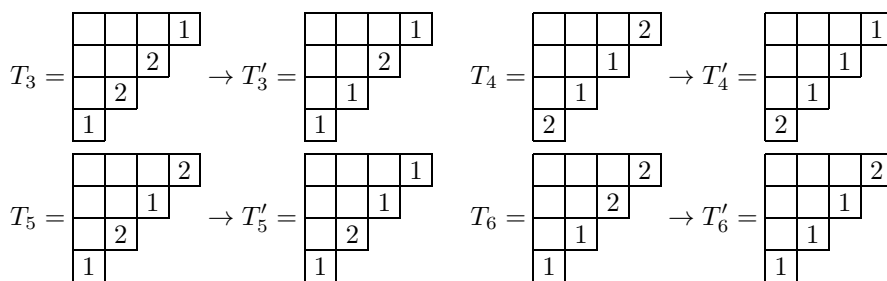


FIGURE 6.3. Correspondence on tableaux.

Lemma 6.1 tells us that under the assumption that $F - D$ is a skew row and a skew column, the LR tableaux count the difference between the ρ_n^F constituents of $\rho_n^D \otimes S^a \otimes S^b$ and the ρ_n^F constituents of $\rho_n^D \otimes S^{a+1} \otimes S^{b-1}$. According to formula (6.1), these constituents must come from $\rho_n^D \otimes \rho_n^{(a,b)}$. Thus the LR formula is correct in this case.

Consider now what happens if $F - D$ is not both a skew row and a skew column. Suppose first that it is a skew row, but not necessarily a skew column. Consider a semistandard tableau T filling $F - D$. We again consider T as a list. We list the entries of T from right to left. Since right to left is compatible with top to bottom, the YWC again guarantees that, at any point in the list, the number of 2's never exceeds the number of 1's up to that point.

We can again think of T as defining an increasing path $\mathcal{P} = \mathcal{P}(T)$ in the plane, from origin to $\begin{bmatrix} a \\ b \end{bmatrix}$. The fact that $F - D$ is not a skew column puts some constraints on what \mathcal{P} can be. We can take the row structure of $F - D$ into account by partitioning the list defining T into subsets consisting of the entries corresponding to boxes in the same row of $F - D$. These will be consecutive subsets of the list T . Call them the *row subsets*. The semistandard condition amounts to requiring that in a row subset the 2's must precede the 1's. If we translate this to the path \mathcal{P} , this becomes the condition, that in a sequence of steps belonging to a row subset, the up moves should precede the right moves. This means that part of the path corresponding to a row subset looks like an upside down L —it has only two legs, with the upward leg preceding the rightward leg. Of course, either leg could have length zero.

Now consider the construction described above, for converting \mathcal{P} to \mathcal{P}' . This proceeds by replacing an upward step by a rightward step. As we noted in the discussion, since it is timed to occur at the first time the highest diagonal is reached, it always happens at a point when an upward step is followed by a rightward step. In other words, it happens at the high point of one of the inverted L 's defined by the row subsets, and it converts such an inverted L to another one, with one less vertical step, and one more rightward step. Thus, the mapping from \mathcal{P} to \mathcal{P}' will take the set of increasing paths from the origin to $\begin{bmatrix} a \\ b \end{bmatrix}$ and satisfying the row-set constraints, to the set of paths from the origin to $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$, and satisfying the row-set constraints. We can also see that the inverse mapping likewise respects the row-set constraints. Thus, the bijection constructed for Lemma 6.1 restricts to define a bijection between the tableaux representing ρ_n^F constituents of $\rho_n^D \otimes S^{a+1} \otimes S^{b-1}$

and the tableaux representing ρ_n^F constituents of $\rho_n^D \otimes S^a \otimes S^b$ that violate the LR condition. Hence, the LR tableaux again count the difference between these two multiplicities, which must be accounted for by the constituents of $\rho_n^D \otimes \rho_n^{(a,b)}$.

Finally, consider the general case, when $F - D$ is not necessarily a skew row. Since it is the union of two skew rows, the columns of $F - D$ are of length at most two. The semistandardness condition gives us no choice about how to fill a column of length two: the upper box must be filled with a 1, and the lower box must be filled with a 2.

Also, the columns of length two occur when a row of the skew row of $E - D$ overlaps a row of the skew row of $F - E$, where E is the intermediate diagram between D and F . Of course, this diagram will vary from tableau to tableau but the overlaps are determined solely by the skew diagram $F - D$. The overlaps always occur on the left end of a row of $E - D$ and on the right end of a row of $F - E$.

If we just delete, or ignore, the columns of F that contain length two columns of $F - D$, then the remaining boxes of $F - D$ form a skew row. If there are c such columns, it requires c 1's and c 2's to fill them. This leaves $(a - c)$ 1's and $(b - c)$ 2's to fill the remaining boxes of $F - D$. Or, if we are looking at the constituents of $\rho_n^D \otimes S^{a+1} \otimes S^{b-1}$, then we would have $(a + 1 - c)$ 1's and $(b - 1 - c)$ 2's to fill the remaining boxes. In other words, after filling the columns of length two in the unique possible way, we are left with a the same problem we considered when $F - D$ was a skew row, except with fewer boxes. Hence, by our previous argument, the LR tableaux count the constituents of $\rho_n^D \otimes S^a \otimes S^b$ that come from $\rho_n^D \otimes \rho_n^{(a,b)}$

The above argument is equivalent to the proof of Littlewood and Richardson for the truth of their rule when the second factor in the tensor product has two rows. When understood on the purely combinatorial level, it does not extend directly to the general case. However, by combining it with the results of [HTW3] and [HL], it can be made to imply the full LR Rule. The constructions of [HTW3] and [HL] give collections of linearly independent highest weight vectors in any tensor product $\rho_n^D \otimes \rho_n^E$, and these collections can be directly seen to be counted by LR tableaux. This implies that the LR numbers are lower bounds for the tensor product multiplicities. The fact that the LR Rule is correct when E has two rows shows that the collections constructed in [HL] span the appropriate space of highest weight vectors for this case, so that they form bases for the relevant spaces. As will be seen in §7, this further implies that the constructions of [HTW3] and [HL] in fact yield bases for all relevant spaces of highest weight vectors, implying the general LR Rule.

7. THE GENERAL CASE

7.1. Recollections from classical invariant theory; the scheme of the proof.

Our proof of the general case of the LR Rule uses classical invariant theory, or more precisely, what in some sense is the core result of classical invariant theory from a modern perspective, the phenomenon of (GL_n, GL_m) -*duality*. We recall it here.

Let $\mathcal{P}(M_{n,m})$ be the algebra of polynomial functions on the space $M_{n,m} = M_{n,m}(\mathbb{C})$ of $n \times m$ complex matrices. We use the standard matrix entries $x_{jk} : 1 \leq j \leq n; 1 \leq k \leq m$ of the typical $n \times m$ matrix X as coordinates on $M_{n,m}$ so that the polynomial functions are sums of the monomials

$$x^\alpha = \prod_{j,k} x_{jk}^{\alpha_{jk}},$$

where α is an $n \times m$ matrix of non-negative integers with entries α_{jk} .

The group $GL_n \times GL_m$ acts on $M_{n,m}$, with GL_n acting by multiplication on the left and GL_m by multiplication on the right. To distinguish the two actions, we shall label GL_m and all its subgroups with a $'$ to indicate that they act by right multiplication. From this we can define an action of $GL_n \times GL'_m$ on $\mathcal{P}(M_{n,m})$ by the formula

$$(7.1) \quad [(g, h).(f)](X) = f(g^t X h),$$

for $g \in GL_n$, $h \in GL'_m$, $X \in M_{n,m}$, and $f \in \mathcal{P}(M_{n,m})$. Here g^t is the transpose of g . (GL_n, GL'_m) duality describes the decomposition of this action into irreducible representations.

Theorem 7.1. *Under the action of $GL_n \times GL'_m$ on $\mathcal{P}(M_{n,m})$ given in equation (7.1), $\mathcal{P}(M_{n,m})$ has the following decomposition into irreducible $GL_n \times GL'_m$ modules:*

$$(7.2) \quad \mathcal{P}(M_{n,m}) \simeq \sum_{r(F) \leq \min(n,m)} \rho_n^F \otimes \rho_m^F.$$

(GL_n, GL'_m) -duality may be regarded as an analog or generalization of the Peter-Weyl Theorem ([BD]) for the unitary group. It has been discovered independently by various authors. See the discussion in [Ho2]. We do not know its first explicit appearance in the literature. For a proof and discussion of its uses in classical invariant theory, see [Ho2] or [GW].

Our first use of (GL_n, GL'_m) -duality is to provide a model for the representations of GL_n and of their tensor products. Recall that $U'_m \subset GL'_m$ is the group of upper triangular unipotent matrices in GL'_m , and A'_m is the subgroup of diagonal matrices. As discussed in §3, the theorem of the highest weight says that for any representation ρ_m^F of GL'_m , the space $(\rho_m^F)^{U'_m}$ of U'_m -invariant vectors is one dimensional and is an eigenspace for A'_m with eigencharacter ψ_m^F . This implies that

$$(7.3) \quad \mathcal{P}(M_{n,m})^{U'_m} \simeq \left(\sum_{r(F) \leq \min(n,m)} \rho_n^F \otimes \rho_m^F \right)^{U'_m} \simeq \sum_{r(F) \leq \min(n,m)} \rho_n^F \otimes (\rho_m^F)^{U'_m}.$$

In other words, the algebra $\mathcal{P}(M_{n,m})^{U'_m}$ of U'_m -invariant functions in $\mathcal{P}(M_{n,m})$ decomposes as a GL_n -module into a sum of one copy of each irreducible representation ρ_n^F . Moreover, since the action of $GL_n \times GL'_m$ and, in particular, the action of A'_m is by algebra automorphisms, the algebra structure is graded by the highest weights ψ_m^F of A'_m , with each A'_m -eigenspace consisting of a single irreducible representation of GL_n .

Variations on this theme let us construct many other related subalgebras. The next example shows how to study the tensor product problem in this context ([HTW3]).

We write $m = k + \ell$, which gives us an isomorphism $M_{n,m} \simeq M_{n,k} \oplus M_{n,\ell}$, where we think of $M_{n,k}$ as defining the first k columns of an $n \times m$ matrix, and $M_{n,\ell}$ as the last ℓ columns. Looking at the polynomial rings, we have

$$(7.4) \quad \mathcal{P}(M_{n,m}) = \mathcal{P}(M_{n,k} \oplus M_{n,\ell}) \simeq \mathcal{P}(M_{n,k}) \otimes \mathcal{P}(M_{n,\ell}).$$

If we apply (GL_n, GL_m) -duality to each of $\mathcal{P}(M_{n,k})$ and $\mathcal{P}(M_{n,\ell})$, and then follow with the construction of formula (7.3), we obtain

$$\begin{aligned}
 \mathcal{P}(M_{n,m})^{U'_k \times U'_\ell} &\simeq \left(\sum_{r(D) \leq \min(n,k)} \rho_n^D \otimes (\rho_k^D)^{U'_k} \right) \otimes \left(\sum_{r(E) \leq \min(n,\ell)} \rho_n^E \otimes (\rho_\ell^E)^{U'_\ell} \right) \\
 (7.5) \quad &\simeq \sum_{D,E} (\rho_n^D \otimes \rho_n^E) \otimes \left((\rho_k^D)^{U'_k} \otimes (\rho_\ell^E)^{U'_\ell} \right).
 \end{aligned}$$

Here we obtain one copy of each possible tensor product of representations of GL_n (subject to some restriction on the depth of the diagrams involved). Again, the algebra is graded, this time by characters $\psi_k^D \times \psi_\ell^E$ of $A'_k \times A'_\ell$. Of course, we can think of $GL'_k \times GL'_\ell \subset GL'_m$ as a block diagonal subgroup, and under this identification, we have $A'_k \times A'_\ell \simeq A'_m$, so the grading of $\mathcal{P}(M_{n,m})^{U'_k \times U'_\ell}$ may be thought of as a grading by \widehat{A}'_m .

We can iterate the process of the previous paragraph, decomposing $M_{n,k}$ or $M_{n,\ell}$ into subsets of columns, and so forth, and obtain corresponding subalgebras of $\mathcal{P}(M_{n,m})$. Thus, we could consider any decomposition $\Gamma = (k_1, \dots, k_c)$ of m into pieces k_i , so that $m = \sum_{i=1}^c k_i$. To this decomposition, we can associate the block diagonal subgroup

$$(7.6) \quad M'_\Gamma = \prod_{i=1}^c GL'_{k_i} \subset GL'_m,$$

where GL'_{k_i} is the subgroup of GL'_m that acts on the variables $x_{a,j}$ for $1 \leq a \leq n$ and $(\sum_{b=1}^{i-1} k_b) + 1 \leq j \leq \sum_{b=1}^i k_b$, and leaves the other variables unchanged. Let

$$(7.7) \quad U'_\Gamma = \prod_{i=1}^c U'_{k_i} \subset U'_m$$

be the group of unipotent upper triangular matrices in M'_Γ . We can look at the subalgebra

$$(7.8) \quad \mathcal{A}(\Gamma) = \mathcal{A}(k_1, \dots, k_c) = \mathcal{P}(M_{n,m})^{U'_\Gamma}.$$

The summands of this algebra will be c -fold tensor products of representations of GL_n , with the depth of the diagram labeling the i th factor being bounded by k_i . More precisely, $\mathcal{A}(\Gamma)$ is a module for $GL_n \times (\prod_{i=1}^c A'_{k_i})$ and can be decomposed as

$$\begin{aligned}
 \mathcal{A}(\Gamma) &\simeq \mathcal{P} \left(\bigoplus_{i=1}^c M_{n,k_i} \right)^{U'_\Gamma} \simeq \bigotimes_{i=1}^c \mathcal{P}(M_{n,k_i})^{U'_{k_i}} \\
 &\simeq \bigotimes_{i=1}^c \left(\sum_{r(D_i) \leq \min(n,k_i)} \rho_n^{D_i} \otimes (\rho_{k_i}^{D_i})^{U'_{k_i}} \right) \\
 (7.9) \quad &\simeq \sum_{D_1, \dots, D_c} (\rho_n^{D_1} \otimes \dots \otimes \rho_n^{D_c}) \otimes (\rho_{k_1}^{D_1})^{U'_{k_1}} \otimes \dots \otimes (\rho_{k_c}^{D_c})^{U'_{k_c}}.
 \end{aligned}$$

Also, this algebra is graded by characters of $\prod_{i=1}^c A'_{k_i} \simeq A'_m$. The subalgebras $\mathcal{A}(\Gamma)$ corresponding to the various decompositions Γ of m have obvious inclusion relations, with one containing another when one decomposition refines another.

The extreme case of this is when $c = m$ and all the k_i are equal to 1. Then all the U'_{k_i} are equal to the identity, and we are just dealing with the full polynomial ring $\mathcal{P}(\mathbb{M}_{n,m})$, decomposed as an m -fold tensor product:

$$(7.10) \quad \mathcal{P}(\mathbb{M}_{n,m}) \simeq \mathcal{P}(\mathbb{C}^n)^{\otimes m}.$$

Of course, the polynomial ring in n variables is decomposed into the subspaces of polynomials of given degree:

$$(7.11) \quad \mathcal{P}(\mathbb{C}^n) = \sum_{d=0}^{\infty} \mathcal{P}^d(\mathbb{C}^n).$$

The spaces

$$(7.12) \quad \mathcal{P}^d(\mathbb{C}^n) \simeq \rho_n^{(d)}$$

are just the symmetric powers of the standard representation of GL_n on \mathbb{C}^n . (It might seem that they should be the duals of the symmetric powers, but we have defined the action of GL_n on $\mathcal{P}(\mathbb{M}_{n,m})$ to be dual to what might seem natural, in order to make equation (7.12) true.) At the same time, of course, the $\mathcal{P}^d(\mathbb{C}^n)$ are also the eigenspaces for the scalar operators, $f(t\vec{v}) = t^d f(\vec{v})$ for $f \in \mathcal{P}^d(\mathbb{C}^n)$. Thus, we obtain the decomposition

$$(7.13) \quad \mathcal{P}(\mathbb{M}_{n,m}) \simeq \sum_{d_i \geq 0} \rho_n^{(d_1)} \otimes \rho_n^{(d_2)} \otimes \dots \otimes \rho_n^{(d_m)}$$

of $\mathcal{P}(\mathbb{M}_{n,m})$ into eigenspaces for $A'_m \simeq (A'_1)^m$, and each eigenspace is described as a tensor product of symmetric powers of \mathbb{C}^n , which are the representations of GL_n labeled by diagrams with one row.

Each of the algebras $\mathcal{A}(k_1, \dots, k_c)$ is invariant under GL_n , so a natural question to ask is, how does $\mathcal{A}(k_1, \dots, k_c)$ decompose as a representation for GL_n ? A strong form of an answer to this question would be provided by a description of the subalgebra $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ of GL_n highest weight vectors in $\mathcal{A}(k_1, \dots, k_c)$. This would give a description of the GL_n highest weight vectors in each summand

$$\rho_n^{D_1} \otimes \dots \otimes \rho_n^{D_c}$$

of formula (7.9), which by highest weight theory determine the GL_n -module structure of this multiple tensor product. In particular, a description of $\mathcal{A}(k, \ell)^{U_n}$ for $k + \ell = m$ would provide an answer to the problem of decomposing tensor products of representations of GL_n . In fact, the algebra $\mathcal{A}(k, \ell)^{U_n}$ is a module for $A_n \times A_k \times A_\ell$ and can be decomposed as

$$(7.14) \quad \mathcal{A}(k, \ell)^{U_n} = \sum_{F,D,E} \mathcal{E}_{F,D,E},$$

where each $\mathcal{E}_{F,D,E}$ is the $\psi_n^F \times \psi_k^D \times \psi_\ell^E$ eigenspace for $A_n \times A_k \times A_\ell$. So the algebra $\mathcal{A}(k, \ell)^{U_n}$ is also graded and the $\mathcal{E}_{F,D,E}$'s are its homogeneous components. The non-zero vectors in $\mathcal{E}_{F,D,E}$ are precisely the GL_n highest weight vectors of weight ψ_n^F in the tensor product $\rho_n^D \otimes \rho_n^E$. Consequently, by equation (3.6), its dimension coincides with the multiplicity of ρ_n^F in $\rho_n^D \otimes \rho_n^E$. Thus information on how tensor products of GL_n representations decompose can be deduced from the structure of the algebra $\mathcal{A}(k, \ell)^{U_n}$. In view of this property, this algebra is called a GL_n tensor product algebra ([HTW3]). Similarly, we call $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ a GL_n c -fold tensor product algebra ([HL]).

Based on the discussion in the previous paragraph, the LR Rule can be formulated as a statement on the dimension of $\mathcal{E}_{F,D,E}$.

Lemma 7.2. *The LR Rule holds if and only if*

$$\dim \mathcal{E}_{F,D,E} = c_{D,E}^F$$

for all Young diagrams D, E , and F .

A description of the tensor product algebra $\mathcal{A}(k, \ell)^{U_n}$ was given in [HTW3], and descriptions of all the algebras $\mathcal{A}(k_1, \dots, k_c)$ were given in [HL]. More precisely, [HTW3] described a collection of elements of $\mathcal{A}(k, \ell)^{U_n}$ that corresponded bijectively to the appropriate LR tableaux, and showed that these elements were linearly independent. Assuming the LR Rule, i.e., that the LR tableaux count the multiplicities in tensor products, we could conclude that the elements of $\mathcal{A}(k, \ell)^{U_n}$ constructed in [HTW3] in fact span $\mathcal{A}(k, \ell)^{U_n}$, so that they form a basis for $\mathcal{A}(k, \ell)^{U_n}$. Similar remarks apply to the more general result of [HL].

Here we will show that how the two-rowed LR Rule provides an alternate route to showing that the elements constructed in [HTW3] in fact span $\mathcal{A}(k, \ell)^{U_n}$, thus providing a proof of the general LR Rule.

Our argument uses various members of the family of algebras $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ and the relationship between them. To study these relationships, it is useful to describe these algebras using a reciprocity phenomenon ([Ho2], [HTW1]). We note that, by (GL_n, GL_m) -duality, we can write

(7.15)

$$\begin{aligned} \mathcal{A}(k_1, \dots, k_c)^{U_n} &= (\mathcal{P}(M_{n,m})^{U'_\Gamma})^{U_n} \simeq \left(\left(\sum_F \rho_n^F \otimes \rho_m^F \right)^{U'_\Gamma} \right)^{U_n} \simeq \left(\sum_F \rho_n^F \otimes (\rho_m^F)^{U'_\Gamma} \right)^{U_n} \\ &\simeq \sum_F (\rho_n^F)^{U_n} \otimes (\rho_m^F)^{U'_\Gamma} \simeq \left(\sum_F (\rho_n^F)^{U_n} \otimes \rho_m^F \right)^{U'_\Gamma} \simeq (\mathcal{P}(M_{n,m})^{U_n})^{U'_\Gamma}. \end{aligned}$$

Thus, after taking the U_n -invariants, we can think of $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ as the algebra of U'_Γ -invariants in the GL'_m module $\sum_F (\rho_n^F)^{U_n} \otimes \rho_m^F$. We note that $U'_\Gamma = \prod_i U'_{k_i}$ is a maximal unipotent upper triangular group in the block diagonal subgroup $M'_\Gamma = \prod_i GL'_{k_i} \subset GL'_m$. Thus, the space $(\rho_m^F)^{U'_\Gamma}$ is the space of the highest weight vectors for M'_Γ in the irreducible representation ρ_m^F of GL'_m . By the theory of the highest weight, the space $(\rho_m^F)^{U'_\Gamma}$ is telling us how ρ_m^F decomposes as an M'_Γ module. Hence the full ring $(\mathcal{P}(M_{n,m})^{U_n})^{U'_\Gamma} \simeq (\sum_F (\rho_n^F)^{U_n} \otimes \rho_m^F)^{U'_\Gamma}$ is telling us how all the representations ρ_m^F decompose on restriction to M'_Γ . That is, it is telling us the *branching rule* from GL'_m to M'_Γ . (This double interpretation of the algebra $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ is part of a general reciprocity phenomenon in the theory of reductive dual pairs in the symplectic group ([HTW1]). For this reason, we also refer to $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ as the (GL'_m, M'_Γ) *branching algebra*.

When $(k_1, \dots, k_c) = (k, 1, 1, \dots, 1) = (k, 1^{m-k})$, with $m - k = \ell$ 1's, the branching algebra $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ turns out to be relatively easy to understand and is closely related to the *standard monomial theory* of Hodge ([Hd]). In the 1990s, standard monomial theory was recast by Gonciulea and Lakschmibai ([GL]) in terms of toric deformations. A similar description was given by Sturmfels ([Stu]), Sturmfels and Miller ([MS]), and Kim ([Kim]) in terms of the highest terms with

respect to an appropriate term order of the elements of $\mathcal{A}(k, 1^{m-k})^{U_n}$. This is the description that will be most useful here, and we will describe it in detail shortly. At the moment, we anticipate that the highest terms of elements of $\mathcal{A}(k, 1^{m-k})^{U_n}$ will parametrize semistandard tableaux in a fairly straightforward way. In this context, the main result of [HTW3] is that there is a collection of elements of $\mathcal{A}(k, \ell)^{U_n}$ whose highest terms correspond to (all of the) Littlewood–Richardson tableaux. In [HL], this result is extended to all algebras $\mathcal{A}(k_1, \dots, k_c)^{U_n}$ for any decomposition $m = \sum_{i=1}^c k_i$.

We now consider decompositions

$$\Xi_{k,i} = (k, \overbrace{1, 1, \dots, 1}^{i-1}, 2, \overbrace{1, 1, \dots, 1}^{\ell-i-1}) = (k, 1^{i-1}, 2, 1^{\ell-i-1}),$$

that start with k , follow with $(i - 1)$ 1’s, then have one 2, and then $m - k - i - 1 = \ell - i - 1$ more 1’s. The algebra $\mathcal{A}(\Xi_{k,i})^{U_n}$ is the subalgebra of $\mathcal{P}(\mathbb{M}_{n,m})^{U_n}$ consisting of functions invariant under the group $U'_k \times U_2'^{(i)}$, where $U_2'^{(i)}$ is the upper unipotent subgroup of $\text{GL}_2'^{(i)}$, which operates on the plane consisting of vectors with only the $k + i$ and $k + i + 1$ coordinates non-zero. The elements of $\mathcal{A}(\Xi_{k,i})^{U_n}$ constructed in [HL] have highest terms correspond to all semistandard tableaux that satisfy the YWC with respect to the pair $(i, i + 1)$. It will be seen that by the Pieri Rule and the two-row case of the LR Rule, these tableaux count the relevant highest weight vectors, and it follows that the elements of $\mathcal{A}(\Xi_{k,i})^{U_n}$ constructed in [HL] form a basis for $\mathcal{A}(\Xi_{k,i})^{U_n}$. It further follows that any function in $\mathcal{A}(\Xi_{k,i})^{U_n}$ must have a highest term that corresponds to a semistandard tableau satisfying the $(i, i + 1)$ YWC. Details will be discussed later.

We now observe that the groups $U_2'^{(i)}$ generate the full group U'_ℓ of upper triangular unipotent matrices in GL'_ℓ . This is easily checked. It follows that, if a function in $\mathcal{P}(\mathbb{M}_{n,m})$ is invariant under each $U_2'^{(i)}$, then it is invariant under U'_ℓ . This implies that

$$\mathcal{A}(k, \ell)^{U_n} = (\mathcal{P}(\mathbb{M}_{n,m})^{U_n})^{U'_k \times U'_\ell} = \bigcap_{i=1}^{\ell-1} (\mathcal{P}(\mathbb{M}_{n,m})^{U_n})^{U'_k \times U_2'^{(i)}} = \bigcap_{i=1}^{\ell-1} \mathcal{A}(\Xi_{k,i})^{U_n}.$$

From the discussion of the previous paragraph, we see that equation (7.16) implies that elements of $\mathcal{A}(k, \ell)^{U_n}$ must have highest terms that correspond to tableaux satisfying the YWC for all pairs $(i, i + 1)$ for i from 1 to $\ell - 1$. On the other hand, we have constructed in [HTW3] an element having any such tableau as a highest term. Thus, the highest weights of representations of GL'_m restricted to $\text{GL}'_k \times \text{GL}'_\ell$ are counted by LR tableau, which is the Littlewood–Richardson Rule.

7.2. The algebra $\mathcal{A}(k, 1^{m-k})^{U_n}$. In the remaining subsections, we will present the details of the arguments sketched in §7.1. This subsection is devoted to the branching algebra $\mathcal{A}(\Gamma)^{U_n} = \mathcal{A}(k, 1^\ell)^{U_n}$, where $k + \ell = m$. For convenience, we shall assume that $k \leq n$. As mentioned above, this is essentially a part of the much-studied standard monomial theory ([GL], [Ho2], [Kim], [MS]), originally due to Hodge ([Hd]). We will describe a basis for the algebra $\mathcal{A}(k, 1^\ell)^{U_n}$. In §7.5, this basis will be used to construct a basis for $\mathcal{A}(\Xi_{k,i})$.

According to equation (7.8),

$$\mathcal{A}(k, 1^\ell)^{U_n} = \left(\mathcal{P}(M_{n,m})^{U'_r} \right)^{U_n} = \mathcal{P}(M_{n,m})^{U_n \times U'_k}$$

because $U'_r = U'_k \times (U'_1)^\ell \simeq U'_k$ since U'_1 is trivial. We restrict the action of $GL_n \times GL'_m$ on $\mathcal{P}(M_{n,m})$ to $GL_n \times (GL'_k \times (GL'_1)^\ell)$, and under this action,

$$\begin{aligned} \mathcal{P}(M_{n,m}) &\simeq \mathcal{P}(M_{n,k} \oplus \mathbb{C}_1^n \oplus \cdots \oplus \mathbb{C}_\ell^n) \\ &\simeq \mathcal{P}(M_{n,k}) \otimes \mathcal{P}(\mathbb{C}_1^n) \otimes \cdots \otimes \mathcal{P}(\mathbb{C}_\ell^n) \\ &\simeq \left(\sum_{r(D) \leq k} \rho_n^D \otimes \rho_k^D \right) \otimes \left(\sum_{\alpha_1 \geq 0} \rho_n^{(\alpha_1)} \otimes \rho_1^{(\alpha_1)} \right) \otimes \cdots \otimes \left(\sum_{\alpha_\ell \geq 0} \rho_n^{(\alpha_\ell)} \otimes \rho_1^{(\alpha_\ell)} \right) \\ &\simeq \sum_{\substack{r(D) \leq k \\ \alpha_1, \dots, \alpha_\ell \geq 0}} \left(\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} \right) \otimes \rho_k^D \otimes \rho_1^{(\alpha_1)} \otimes \cdots \otimes \rho_1^{(\alpha_\ell)}. \end{aligned}$$

Here, for each $1 \leq j \leq m - k$, \mathbb{C}_j^n is a copy of \mathbb{C}^n . By extracting the $U_n \times U'_k$ invariants from $\mathcal{P}(M_{n,m})$, we obtain

$$\begin{aligned} \mathcal{A}(k, 1^\ell)^{U_n} &= \mathcal{P}(M_{n,m})^{U_n \times U'_k} \\ &\cong \sum_{\substack{r(D) \leq k \\ \alpha_1, \dots, \alpha_\ell \geq 0}} \left(\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} \right)^{U_n} \\ &\quad \otimes \left(\rho_k^D \right)^{U'_k} \otimes \rho_1^{(\alpha_1)} \otimes \cdots \otimes \rho_1^{(\alpha_\ell)}. \end{aligned}$$

The algebra $\mathcal{A}(k, 1^\ell)^{U_n}$ is a module for $A_n \times A'_k \times (GL'_1)^\ell$. We will identify $(GL'_1)^\ell$ with the diagonal torus of GL'_ℓ , and the representation $\rho_1^{(\alpha_1)} \otimes \cdots \otimes \rho_1^{(\alpha_\ell)}$ of $(GL'_1)^\ell$ with the character ψ_ℓ^α of A'_ℓ , where $\alpha = (\alpha_1, \dots, \alpha_\ell)$. Now $\mathcal{A}(k, 1^\ell)^{U_n}$ can be decomposed as

$$\mathcal{A}(k, 1^\ell)^{U_n} = \sum_{F,D,\alpha} W_{F,D,\alpha},$$

where the sum is taken over all Young diagrams D and F with $r(D) \leq k$ and $r(F) \leq n$ and over all $\alpha \in \mathbb{Z}_{\geq 0}^\ell$, and $W_{F,D,\alpha}$ is the $\psi_n^F \times \psi_k^D \times \psi_\ell^\alpha$ -eigenspace of $A_n \times A'_k \times A'_\ell$. If $W_{F,D,\alpha}$ is non-zero, then the non-zero vectors in $W_{F,D,\alpha}$ can be identified with the GL_n highest weight vectors of weight ψ_n^F in the tensor product

$$(7.17) \quad \rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)}.$$

So the dimension of $W_{F,D,\alpha}$ coincides with the multiplicity of ρ_n^F in the tensor product (7.17). Let $ST(F, D, \alpha)$ be the set of all semistandard tableaux of shape $F - D$ and content α . Then by the Pieri Rule discussed in §5, this multiplicity coincides with the cardinality of $ST(F, D, \alpha)$. Thus

$$(7.18) \quad \dim W_{F,D,\alpha} = \#(ST(F, D, \alpha)).$$

We shall construct a basis for $W_{F,D,\alpha}$ labeled by the elements of $ST(F, D, \alpha)$. For these purposes, we write a typical element of $M_{n,m}$ as

$$(7.19) \quad \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} & y_{11} & y_{12} & \cdots & y_{1\ell} \\ x_{21} & x_{22} & \cdots & x_{2k} & y_{21} & y_{22} & \cdots & y_{2\ell} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} & y_{n1} & y_{n2} & \cdots & y_{n\ell} \end{pmatrix},$$

so that $\mathcal{P}(M_{n,m})$ can be viewed as the polynomial algebra on the variables x_{ab} and y_{cd} . We want to define elements of $\mathcal{A}(k, 1^\ell)^{U_n}$ corresponding to column skew tableaux. A column skew tableau is defined by a certain number p of empty boxes at the top, followed by boxes labeled by an increasing sequence $S = \{s_1, s_2, \dots, s_q\}$ of whole numbers s_j with $s_j < s_{j+1}$. Let us label this column skew tableau $T_{(p,S)}$. For every column tableau $T_{(p,S)}$ with $p \leq k$ and $s_q \leq \ell$, we define an element of the algebra $\mathcal{A}(k, 1^\ell)^{U_n}$ by the formula

$$(7.20) \quad \gamma_{(p,S)} = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1p} & y_{1s_1} & y_{1s_2} & \cdots & y_{1s_q} \\ x_{21} & x_{22} & \cdots & x_{2p} & y_{2s_1} & y_{2s_2} & \cdots & y_{2s_q} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{(p+q)1} & x_{(p+q)2} & \cdots & x_{(p+q)p} & y_{(p+q)s_1} & y_{(p+q)s_2} & \cdots & y_{(p+q)s_q} \end{vmatrix}.$$

When $q = 0$, the sequence S is empty, in which case

$$\gamma_{(p,\emptyset)} = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{vmatrix}.$$

Similarly, if $p = 0$, then

$$\gamma_{(0,S)} = \begin{vmatrix} y_{1s_1} & y_{1s_2} & \cdots & y_{1s_q} \\ y_{2s_1} & y_{2s_2} & \cdots & y_{2s_q} \\ \vdots & \vdots & & \vdots \\ y_{qs_1} & y_{qs_2} & \cdots & y_{qs_q} \end{vmatrix}.$$

It is easy to check that because the determinant (7.20) involves the first $p + q$ consecutive rows of the matrix (7.19), $\gamma_{(p,S)}$ is a highest weight vector (of weight $\psi_n^{1_{p+q}}$) for GL_n . See equation (3.4) for the notation 1_{p+q} . Similarly, since the determinant (7.20) involves the first p consecutive columns of the x_{ab} , $\gamma_{(p,S)}$ will also be a highest weight vector (of weight $\psi_k^{1_p}$) for GL'_k . Also, it will be a weight vector for the torus $A'_\ell \subset GL'_\ell$ with weight $\psi'_\ell^{\alpha(S)}$, where $\alpha(S)$ is the ℓ -tuple with entries 1 at the places belonging to S , and 0's at the other places. Thus, we see that

$$(7.21) \quad \gamma_{(p,S)} \in W_{\mathbf{1}_{p+q}, \mathbf{1}_p, \alpha(S)}.$$

Now consider any semistandard skew tableau T . Let the j th column (counting from left to right) of T be $T_j = T_{(p_j, S_j)}$. Assume that $p_j \leq k$ and that the largest entry of S_j is at most ℓ . We define the *standard monomial* corresponding to T as

$$(7.22) \quad \gamma_T = \prod_{j=1}^c \gamma_{T_j} = \prod_{j=1}^c \gamma_{(p_j, S_j)},$$

where c is the number of columns in T . The main result of standard monomial theory ([Hd], [GL], [MS]) is

Theorem 7.3. *The set $\mathcal{B}_{F,D,\alpha} = \{\gamma_T : T \in \text{ST}(F, D, \alpha)\}$ is a basis for $W_{F,D,\alpha}$.*

Since we have constructed an element γ_T of the vector space $W_{F,D,\alpha}$ for each tableau T in $\text{ST}(F, D, \alpha)$, the theorem will follow if we can show that the γ_T are linearly independent as T varies over $\text{ST}(F, D, \alpha)$. We will do this by introducing

a monomial ordering τ for $\mathcal{P}(M_{n,m})$. That is, τ is a total ordering on the set of monomials in the entries of the matrix (7.19), and satisfying certain basic properties, as detailed in, for example [CLO].

We define τ by first putting a linear ordering on the variables x_{ij} and y_{is} . Here $1 \leq i \leq n, 1 \leq j \leq k$, and $1 \leq s \leq \ell$. We agree that

$$(7.23) \quad \begin{aligned} & \text{(a) } x_{ab} > x_{cd} \text{ iff } b < d \text{ or } b = d \text{ and } a < c. \\ & \text{(b) Similarly, } y_{ab} > y_{cd} \text{ iff } b < d \text{ or } b = d \text{ and } a < c. \end{aligned}$$

Thus,

$$x_{11} > x_{21} > \cdots > x_{n1} > x_{12} > x_{22} > \cdots > x_{n-1,k} > x_{nk},$$

and similarly for the y_{is} .

$$(c) \text{ Finally, } x_{ab} > y_{cd} \text{ for all pairs } (a, b) \text{ and } (c, d) \text{ of indices.}$$

We then extend this ordering to a total ordering on all monomials in the x_{ij} and y_{is} by the graded lexicographic ordering (see [CLO]). This is described as follows. Let N be an $n \times k$ matrix of non-negative integers n_{ij} , and let M be an $n \times \ell$ matrix of non-negative integers m_{is} . Then a typical monomial in the x_{ij} and y_{is} can be written as

$$(7.24) \quad x^N y^M = \left(\prod_{i,j} x_{ij}^{n_{ij}} \right) \left(\prod_{s,t} y_{st}^{m_{st}} \right).$$

The graded lexicographic order associated to the ordering (7.23) of the variable says that, given two monomials $x^{N_1} y^{M_1}$ and $x^{N_2} y^{M_2}$, we have

$$(7.25) \quad x^{N_1} y^{M_1} > x^{N_2} y^{M_2}$$

if either

- i) the degree of $x^{N_1} y^{M_1}$ is larger than the degree of $x^{N_2} y^{M_2}$; or
- ii) both $x^{N_1} y^{M_1}$ and $x^{N_2} y^{M_2}$ have the same degree, and for the first (i.e., largest) variable with an exponent in $x^{N_1} y^{M_1}$ that is different from its exponent in $x^{N_2} y^{M_2}$, the exponent in $x^{N_1} y^{M_1}$ is larger.

Definition 7.4. Let f be a polynomial in $\mathcal{P}(M_{n,m})$. Since f is a finite linear combination of the monomials $x^N y^M$, and since τ is a total ordering, among the monomials appearing in f with non-zero coefficients, there will be a maximal one. We call this the leading monomial of f , and denote it by $\text{in}_\tau(f)$.

We recall ([CLO]) that a basic fact about leading monomials is that leading monomials are compatible with multiplication: if f and ϕ are two polynomials in $\mathcal{P}(M_{n,m})$

$$\text{in}_\tau(f\phi) = \text{in}_\tau(f)\text{in}_\tau(\phi).$$

For each skew tableau T of shape $F - D$, define

$$(7.26) \quad m_{(D,F),T} = \left(\prod_{i=1}^k x_{ii}^{\lambda_i} \right) \left(\prod_{\mathbf{b} \in T} y_{a(\mathbf{b})c(\mathbf{b})} \right),$$

where $D = (\lambda_1, \dots, \lambda_k)$, and for each box \mathbf{b} in T , $a(\mathbf{b})$ is the row of F in which the box \mathbf{b} lies, and $c(\mathbf{b})$ is the entry in \mathbf{b} . For example, if D, F and T are given in equation (4.1), then

$$m_{(D,F),T} = x_{11}^3 x_{22}^2 x_{33} y_{11}^2 y_{12} y_{22} y_{23} y_{31} y_{33} y_{34} y_{41} y_{42}.$$

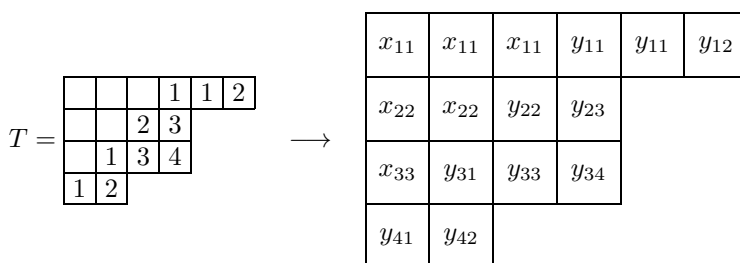


FIGURE 7.1. Fill the boxes of a tableau with variables

Here is another way of thinking about $m_{(D,F),T}$. For each box of T , fill the box with a variable. If the box is empty (i.e., is a box of D), then fill it with x_{ii} , where i is the row in which the box occurs. If the box contains the number s and is in row i , then fill it with the variable y_{is} . Then $m_{(D,F),T}$ is the product of all the variables filling the boxes of T . For the example above, we fill the boxes of T as shown in Figure 7.1.

From this description, it is evident that the exponent λ_i with which x_{ii} appears in $m_{(D,F),T}$ is exactly the length of the i th row of D and the exponent c_{it} with which y_{it} appears is the number of t s in row i . This makes it clear that a tableaux T can be recovered from $m_{(D,F),T}$. Indeed, since the boxes in any row of T are arranged with the empty boxes on the left and the entries increasing from left to right, knowing the λ_i and the numbers c_{it} of boxes filled by t in each row i is enough to reconstitute T .

Lemma 7.5. For each $T \in \text{ST}(F, D, \alpha)$,

$$(7.27) \quad \text{in}_\tau(\gamma_T) = m_{(D,F),T}.$$

Proof. Let $T_j = T_{(p_j, S_j)}$, $1 \leq j \leq c$, be the columns of T , so that $\gamma_T = \prod_{j=1}^c \gamma_{(p_j, S_j)}$. Since leading monomials are multiplicative,

$$\text{in}_\tau(\gamma_T) = \text{in}_\tau \left(\prod_{j=1}^c \gamma_{(p_j, S_j)} \right) = \prod_{j=1}^c \text{in}_\tau(\gamma_{(p_j, S_j)}).$$

On the other hand, $m_{(D,F),T}$ is also multiplicative over the columns of T ; that is,

$$m_{(D,F),T} = \prod_{j=1}^c m_{(D_j, F_j), T_j},$$

where each D_j are F_j are column diagrams with p_j and $p_j + q_j$ boxes respectively, and $q_j = \#(S_j)$. Hence it is enough to check formula (7.27) for a column tableau $T = T_{(p, S)}$ where $S = \{s_1, \dots, s_q\}$ with $s_j < s_{j+1}$. In this case, $\gamma_T = \gamma_{(p, S)}$ which is the determinant given in equation (7.20). To determine its leading monomial, we observe that x_{11} has the highest order among all the variables which appear in the determinant. From the definition of the monomial ordering τ given in (7.23), we see that $\text{in}_\tau(\gamma_T)$ must contain the variable x_{11} . To determine the remaining variables in $\text{in}_\tau(\gamma_T)$, we consider the determinant of the submatrix obtained by removing the first row and first column from the original matrix. Among all the variables which appear in this submatrix, x_{22} has the highest order. So $\text{in}_\tau(\gamma_T)$ must also contain

the variable x_{22} . Continuing this way, we see that $\text{in}_\tau(\gamma_T)$ is given by the product of all the variables on the diagonal, that is, $\text{in}_\tau(\gamma_T) = x_{11} \cdots x_{pp} y_{(p+1)s_1} \cdots y_{(p+q)s_q}$. This coincides with $m_{(D,F),T}$. \square

Proof of Theorem 7.3. By Lemma 7.5, the leading monomials $\text{in}_\tau(\gamma_T)$ of γ_T for $T \in \text{ST}(F, D, \alpha)$ are distinct. Hence $\mathcal{B}_{F,D,\alpha}$ is linearly independent. \square

Corollary 7.6. *If f is a non-zero element of $W_{F,D,\alpha}$, then*

$$\text{in}_\tau(f) = m_{(D,F),T}$$

for some $T \in \text{ST}(F, D, \alpha)$.

Proof. Since f is a non-zero element of $W_{F,D,\alpha}$, it is a linear combination of vectors in the basis $\mathcal{B}_{F,D,\alpha}$. Since the leading monomials of the vectors in $\mathcal{B}_{F,D,\alpha}$ are distinct, $\text{in}_\tau(f)$ coincides with the leading monomial of one of the basis vector which appears in the linear combination. This shows that $\text{in}_\tau(f)$ is of the desired form. \square

Remark. a) The statement of Theorem 7.3 is more or less as Hodge made it ([Hd]). (The context of Theorem 7.3 is somewhat more general than Hodge considered, but is handled by the same methods.) However, our proof shows much more, which we will explain here.

Hodge’s result inspired numerous authors (e.g., [Stu], [DEP], [Hi], [Se], [GL], [KM], [MS], [Kim]) to attempt to understand it in a more structural way. Since the 1990s, it has come to be understood as a statement that the algebra $\mathcal{A}(k, 1^\ell)^{U_n}$ has a “toric deformation”, a flat deformation ([EH]) to a certain semigroup ring. Both the existence of the deformation and the form of the semigroup ring are of interest. For this reason, although it is not essential for finishing the proof of the LR Rule, we will take some time to describe them. These remarks will be somewhat lengthy, and the reader may wish to skip them on a first reading.

b) Our first observation is that Theorem 7.3 shows that $\mathcal{A}(k, 1^\ell)^{U_n}$ contains a collection of large polynomial subalgebras. In fact, it singles out a finite collection of polynomial subrings that together span the whole algebra. To see this, we note that one can put a partial order on the column tableaux. Given two column tableaux, T_1 and T_2 , we will say

$$(7.28) \quad T_1 \leq T_2$$

provided that the entry in each box of T_1 is less than or equal to the entry in the box of T_2 in the same row, with the conventions that

- i) an empty box is less than any numbered box; and
- ii) an empty space (no box at all) is larger than any box. In other words, $T_1 \leq T_2$ if T_1 is at least as long as T_2 , and for each non-empty box of T_1 for which T_2 has a box in the same row, the entry in that box is at least as large as the entry in the given box of T_1 .

Note that this order already appeared in §2.3 for the case of columns of equal length (cf. (2.4)).

It is easy to convince oneself that, given this definition, a tableau is semistandard if and only if its columns, read left to right, form a weakly increasing sequence with respect to this partial order. Put another way, if we have a linearly ordered sequence of columns, then listing them in order, with arbitrary repetitions allowed,

will produce a semistandard tableau, and all semistandard tableaux arise in this way.

Thus, in view of our definition of standard monomials, we see that every standard monomial arises as a monomial in elements $\gamma_{T_i} = \gamma_{(p_i, S_i)}$, where the T_i form a linearly ordered set. The fact that these elements form a basis for $\mathcal{A}(k, 1^\ell)^{U_n}$ implies in particular that all the monomials formed from a given linearly ordered set of columns are linearly independent. In other words, the generators corresponding to any linearly ordered set of monomials form a polynomial ring. Furthermore, since any standard monomial belongs to at least one such ring, these polynomial rings span $\mathcal{A}(k, 1^\ell)^{U_n}$. Also, it is clear that the intersection of any two such rings is just the polynomial ring in the generators that are in the intersection of the two linearly ordered generating sets. In considering these rings, clearly it is enough to restrict attention to maximal linearly ordered sets. Non-maximal sets will generate subrings of the rings generated by any maximal (linearly ordered) set containing it. We describe this situation by saying that the algebra $\mathcal{A}(k, 1^\ell)^{U_n}$ is an *almost direct sum* of the polynomial rings generated by maximal linearly ordered families of column tableaux under the ordering (7.28).

This is the first lesson of standard monomial theory. It clearly provides insight into the structure of $\mathcal{A}(k, 1^\ell)^{U_n}$, but it leaves in question how the various polynomial subrings fit together inside the whole ring. We turn to that question.

c) Consider a space \mathbb{C}^r , with variables z_j , for $1 \leq j \leq r$. It is clear from the multiplicative property of leading monomials, that if τ is any term order on the monomials in the z_j and B is a subalgebra of the polynomial ring $\mathcal{P}(\mathbb{C}^r)$, then the set $\text{in}_\tau(B)$ of leading monomials of elements of B constitute a semigroup, and their linear span $\mathbb{C}(\text{in}_\tau(B))$ will be the corresponding semigroup ring.

Assume that $\text{in}_\tau(B)$ is finitely generated. Then $\mathbb{C}(\text{in}_\tau(B))$ will be a Noetherian ring, and will be the ring of regular functions on an algebraic variety $V(\text{in}_\tau(B))$. The group of A_r of diagonal matrices on \mathbb{C}^r acts on the algebra $\mathcal{P}(\mathbb{C}^r)$ in the usual way, and the monomials are the eigenfunctions for this action. In particular, $\text{in}_\tau(B)$ and $\mathbb{C}(\text{in}_\tau(B))$ will be invariant under A_r , and so A_r acts on the variety $V(\text{in}_\tau(B))$. Since A_r acts on \mathbb{C}^r with an open orbit, the same will be true of $V(\text{in}_\tau(B))$. This means that $V(\text{in}_\tau(B))$ is what is known as a *toric variety* [Fu3].

Moreover, the ring $\mathbb{C}(\text{in}_\tau(B))$ is closely related to the original ring B , and $V(\text{in}_\tau(B))$ is likewise related to the varieties $V(B)$ associated to B . The precise relationship is captured by the notion of flat deformation: the ring $\mathbb{C}(\text{in}_\tau(B))$ is a flat deformation of B [CHV], and the same is said of the corresponding varieties. Since $V(\text{in}_\tau(B))$ is a toric variety, one speaks of a *toric deformation* of B or of $V(B)$.

d) For the ring $\mathcal{A}(k, 1^\ell)^{U_n}$ and the monomial order τ defined above, we know from Corollary 7.6 that $\text{in}_\tau(\mathcal{A}(k, 1^\ell)^{U_n})$ consists of the monomials $m_{(D,F),T}$. These monomials therefore form a semigroup. In the context of Remark b), we would like to know whether $\text{in}_\tau(\mathcal{A}(k, 1^\ell)^{U_n})$ is finitely generated. It is, and in fact, it has an interesting structure, which we will describe.

As we saw from the description of $m_{(D,F),F}$ given after equation (7.26), $\text{in}_\tau(\mathcal{A}(k, 1^\ell)^{U_n})$ can be identified with the collection of pairs (λ, C) , where λ is the column vector of row lengths of the diagram D , and C is the $n \times \ell$ matrix with entries c_{it} , where c_{it} is the number of t 's in the i th row of the tableaux T .

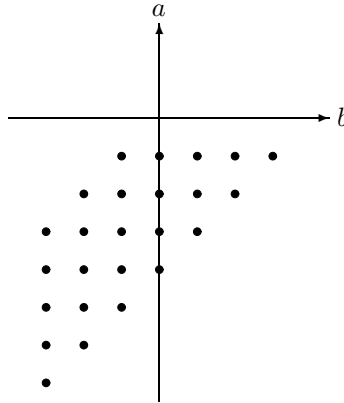


FIGURE 7.2. An illustration of $GT_{(7,3,4)}$

The conditions defining semistandard tableaux can easily be expressed by inequalities among sums of the components of (λ, C) . For convenience in describing this, we set $\lambda_i = c_{i0}$. (Note then that $c_{i0} = 0$ for $i > k$.) Let D_s be the diagram consisting of boxes of T that are filled with numbers up to s . Then the length of the i th row of the diagram D_s is the sum

$$(7.29) \quad r_{is} = \sum_{t=0}^s c_{it}.$$

The condition that the lengths of the rows of D_s increase (weakly) as s increases, in other words, the condition that $r_{is} - r_{i(s-1)} = c_{is} \geq 0$, can be captured by the inequality

$$(7.30) \quad r_{is} \leq r_{i(s+1)}$$

for $s \geq 0$. The semistandardness condition, that the numbers increase strictly down columns, can be expressed similarly: it requires that the length of the i th row of D_s be not longer than the $(i - 1)$ -th row of D_{s-1} :

$$(7.31) \quad r_{(i-1)(s-1)} \geq r_{is}.$$

(Note that this is stronger than saying that D_s has weakly decreasing row lengths.) The condition that D should have at most k rows translates to $c_{(k+1)0} = r_{(k+1)0} = 0$, which from (7.29) and (7.30) implies that also $r_{(k+a)b} = 0$ for $0 \leq b < a$. It is not hard to show that, conversely, any collection of numbers r_{is} satisfying inequalities (7.30) and (7.31), and the additional condition coming from a limitation on k , come from a skew diagram of the appropriate sort.

It turns out that there is an elegant way to characterize the collections of numbers r_{is} satisfying (7.30) and (7.31), plus the side condition. Consider the set of points

$$(7.32) \quad \begin{aligned} GT_{(n,k,\ell)} &= \left\{ \begin{bmatrix} t-i \\ -i \end{bmatrix} : 1 \leq i \leq n, 1 \leq t \leq \ell, i-t \leq k \right\} \\ &= \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : -n \leq b \leq -1, b \leq a \leq b + \ell, a \geq -k \right\} \end{aligned}$$

in the lattice \mathbb{Z}^2 in the plane. The set $GT_{(n,k,\ell)}$ is illustrated in Figure 7.2.

We consider \mathbb{Z}^2 to be a partially ordered set (poset) with respect to the usual partial order

$$\begin{bmatrix} a \\ b \end{bmatrix} \leq \begin{bmatrix} c \\ d \end{bmatrix}$$

if and only if $a \leq c$ and $b \leq d$, and we consider $\text{GT}_{(n,k,\ell)}$ to be a poset with the induced partial ordering.

Given a collection of numbers r_{it} as in inequalities (7.30) and (7.31), define a function R on $\text{GT}_{(n,k,\ell)}$ by the recipe

$$(7.33) \quad R \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = r_{(-b)(a-b)}.$$

It is easy to check that R is an increasing function on the poset $\text{GT}_{(n,k,\ell)}$, and conversely, that every collection r_{it} comes from an increasing function on $\text{GT}_{(n,k,\ell)}$. Patterns such as the r_{it} were defined by Gelfand and Tsetlin ([GT]) as a means of describing bases for representations of GL_n without formally introducing the underlying poset. Hence we call $\text{GT}_{(n,k,\ell)}$ a *Gelfand-Tsetlin poset*. In summary, the semigroup $\text{in}_\tau(\mathcal{A}(k, 1^\ell)^{U_n})$ is isomorphic to the semigroup of \mathbb{Z}^+ -valued, increasing functions on the Gelfand-Tsetlin poset, $\text{GT}_{(n,k,\ell)}$.

How does this relate to the observations of Remark b)? For any poset X , the semigroup $\mathbb{Z}_\geq^+(X)$ consisting of increasing \mathbb{Z}^+ -valued functions on X is a particularly nice kind of semigroup. In particular, it is a lattice cone: the set of integral points in a the cone $\mathbb{R}_\geq^+(X)$ of non-negative, real-valued, order-preserving functions on X , inside the real vector space \mathbb{R}^X of all real-valued functions on X .

The semigroup rings $\mathbb{C}(\mathbb{Z}_\geq^+(X))$ were studied by T. Hibi ([Hi]), and are known as *Hibi rings*. Thus, one way to summarize the derivation of standard monomial theory given above is to say that the algebra $\mathcal{A}(k, 1^\ell)^{U_n}$ has a flat deformation to the Hibi ring $\mathbb{C}(\mathbb{Z}_\geq^+(\text{GT}_{(n,k,\ell)}))$. This is stronger than Hodge's original result. It is due to a number of authors, starting with [GL] and [Stu] in the 1990s, and continuing with [KM], [MS], [Kim] in the 2000s.

It turns out that the Hibi lattice cones $\mathbb{Z}_\geq^+(X)$ are especially nice semigroups. In particular, any Hibi ring has a “standard monomial theory”, in the sense that $\mathbb{C}(\mathbb{Z}_\geq^+(X))$ is the almost direct sum of polynomial rings in a canonical way. Precisely, the generators of $\mathbb{Z}_\geq^+(X)$ are the characteristic functions of increasing subsets of X , and any linearly ordered family of increasing subsets will generate a polynomial ring inside $\mathbb{C}(\mathbb{Z}_\geq^+(X))$. The maximal such families are in one-to-one correspondence with the total orderings on X that are compatible with the given partial ordering. In this context, the column tableaux are in natural bijection with the increasing subsets of $\text{GT}_{(n,k,\ell)}$. We refer to [Ho3] for a more complete discussion. See also [Rei].

e) Since the proof given above of standard monomial theory for $\mathcal{A}(k, 1^\ell)^{U_n}$ establishes a bijection between semistandard tableaux and the semigroup $\mathbb{Z}_\geq^+(\text{GT}_{(n,k,\ell)})$, it follows that semistandard tableaux can be endowed with a semigroup structure. Although this structure does not leap out at one from the yoga of semistandard tableaux, it is not hard to see how to define it. Given two semistandard tableaux T_1 and T_2 , their sum $T_1 + T_2$ is the *concatenation* of T_1 and T_2 , gotten by combining the i th rows of T_1 and T_2 and rearranging the boxes to put them in increasing order, for each $i \geq 1$.

f) We can also use the highest terms of $\mathcal{A}(k, 1^\ell)^{U_n}$ to discuss the YWC. Since c_{it} tells the number of t 's in row i , the YWC can be expressed as the requirement

$$(7.34) \quad \sum_{j=1}^{i-1} c_{j(s-1)} \geq \sum_{j=1}^i c_{js}.$$

The condition is parallel to the strict-decrease-down-rows condition (7.31). The difference is that it involves a sum on the first index rather than the second index of the c_{it} . Because of this, the YWC is not easily expressed in terms of the variables r_{is} that are used to identify the Hibi cone structure on $\text{in}_\tau(\mathcal{A}(k, 1^\ell)^{U_n})$.

On the other hand, this does make clear that the Littlewood-Richardson tableaux form a subcone of $\text{in}_\tau(\mathcal{A}(k, 1^\ell)^{U_n})$; we call it the *LR cone*. Each set of conditions on the c_{ij} , the semistandard conditions, or the YWC conditions would by themselves define a Hibi cone. However, since the two sets of conditions are not compatible, the structure of the LR cone is substantially more complicated than that of Hibi cones. Finding a good model for the LR cone has been the subject of several papers [BZ], [KT1], [KTW], and was an important ingredient in the Knutson-Tau proof of saturation, used to complete the proof of Horn's Conjecture.

7.3. The algebra $\mathcal{A}(\Xi_{k,i})^{U_n}$. We first recall the construction of the algebra $\mathcal{A}(\Xi_{k,i})$. Consider the action of $\text{GL}_n \times \text{GL}'_m$ on $\mathcal{P}(M_{n,m})$ given in formula (7.1), and restrict it to $\text{GL}_n \times (\text{GL}'_k \times M')$ where

$$M' = M'_{\Xi_{k,i}} = \text{GL}'_k \times (\text{GL}'_1)^{i-1} \times \text{GL}'_2 \times (\text{GL}'_1)^{\ell-i-1} \subset \text{GL}'_\ell.$$

Then (see formula (7.8))

$$\mathcal{A}(\Xi_{k,i}) = \mathcal{P}(M_{n,m})^{U'},$$

where $U' = U'_{\Xi_{k,i}} = U'_k \times (U'_1)^{i-1} \times U'_2 \times (U'_1)^{\ell-i-1}$. It is a module for

$$\text{GL}_n \times A'_k \times (A'_1)^{i-1} \times A'_2 \times (A'_1)^{\ell-i-1} \simeq \text{GL}_n \times A'_k \times A'_\ell,$$

and by equation (7.9), it can be decomposed as

$$\begin{aligned} \mathcal{A}(\Xi_{k,i}) &\simeq \sum_{D,\alpha} \left\{ \rho_n^D \otimes \left(\bigotimes_{p=1}^{i-1} \rho_n^{(\alpha_p)} \right) \otimes \rho_n^{(\alpha_i, \alpha_{i+1})} \otimes \left(\bigotimes_{q=i+2}^\ell \rho_n^{(\alpha_q)} \right) \right\} \\ &\quad \otimes (\rho_k^D)^{U'_k} \otimes \left(\bigotimes_{p=1}^{i-1} \rho_1^{(\alpha_p)} \right) \otimes (\rho_2^{(\alpha_i, \alpha_{i+1})})^{U'_2} \otimes \left(\bigotimes_{q=i+2}^\ell \rho_1^{(\alpha_q)} \right) \\ &\simeq \sum_{D,\alpha} \left\{ \rho_n^D \otimes \left(\bigotimes_{p=1}^{i-1} \rho_n^{(\alpha_p)} \right) \otimes \rho_n^{(\alpha_i, \alpha_{i+1})} \otimes \left(\bigotimes_{q=i+2}^\ell \rho_n^{(\alpha_q)} \right) \right\} \otimes \psi_k^D \otimes \psi_\ell^\alpha, \end{aligned}$$

where the sum is taken over all Young diagrams D with $r(D) \leq k$ and all $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ such that $\alpha_i \geq \alpha_{i+1}$.

Now $\mathcal{A}(\Xi_{k,i})^{U_n}$ is a module for $A_n \times A'_k \times A'_\ell$, and it can be decomposed as

$$\mathcal{A}(\Xi_{k,i})^{U_n} = \sum_{F,D,\alpha} \mathcal{E}_{F,D,\alpha}^{(i)},$$

where the sum is taken over all Young diagrams D and F with $r(D) \leq k$ and $r(F) \leq m$, and all $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ such that $\alpha_i \geq \alpha_{i+1}$, and $\mathcal{E}_{F,D,\alpha}^{(i)}$ is the $\psi_n^F \times \psi_n^D \times \psi_\ell^\alpha$ eigenspace for $A_n \times A'_k \times A'_\ell$. If $\mathcal{E}_{F,D,\alpha}^{(i)}$ is non-zero, then the non-zero vectors in $\mathcal{E}_{F,D,\alpha}^{(i)}$ can be identified with the GL_n highest weight vectors of weight ψ_n^F in the tensor product

$$(7.35) \quad \rho_n^D \otimes \left(\bigotimes_{p=1}^{i-1} \rho_n^{(\alpha_p)} \right) \otimes \rho_n^{(\alpha_i, \alpha_{i+1})} \otimes \left(\bigotimes_{q=i+2}^\ell \rho_n^{(\alpha_q)} \right).$$

So the dimension of $\mathcal{E}_{F,D,\alpha}^{(i)}$ coincides with the multiplicity of ρ_n^F in the tensor product (7.35). We now describe this multiplicity.

Recall that a skew tableau is an LR tableau if it is semistandard and it satisfies the YWC. The YWC is a condition on the pairs $(i, i + 1)$ for $1 \leq i \leq \ell - 1$, but it can also be viewed as a set of $(\ell - 1)$ distinct conditions with each condition on a specific pair. We now consider those semistandard tableaux in which the YWC for a specific pair is satisfied.

Notation. Fix $1 \leq i \leq \ell - 1$. Let D and F be Young diagrams such that $r(D) \leq k$ and $r(F) \leq n$, and let $\alpha = (\alpha_1, \dots, \alpha_\ell)$ be an ℓ -tuple of non-negative integers such that $\alpha_i \geq \alpha_{i+1}$. Let $ST^{(i)}(F, D, \alpha)$ be the set of all semistandard tableaux T with the following properties:

- (i) T is of shape $F - D$ and has content α .
- (ii) The pair $(i, i + 1)$ satisfies the YWC in T . This means that starting from the first entry of $w(T)$ to any place in $w(T)$, there are at least as many i 's as $(i + 1)$'s.

Lemma 7.7. *The dimension of $\mathcal{E}_{F,D,\alpha}^{(i)}$ coincides with the cardinality of $ST^{(i)}(F, D, \alpha)$.*

Proof. By the Pieri Rule and the LR Rule for the two-row case, the multiplicity of ρ_n^F in the tensor product (7.35) is given by the number of all $(\ell - 1)$ -tuples $(F_1, \dots, F_{\ell-2}, T_i)$ where

- (i) $D = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_{\ell-2} \subset F = F_{\ell-1}$ is a chain of Young diagrams.
- (ii) For each $1 \leq j \leq i - 1$, $F_j - F_{j-1}$ is a skew row with α_j boxes.
- (iii) T_i is a LR tableau of shape $F_i - F_{i-1}$ and content (α_i, α_{i+1}) .
- (iv) For each $i + 1 \leq j \leq \ell - 1$, $F_j - F_{j-1}$ is a skew row with α_{j+1} boxes.

We denote the collections of all such $(\ell - 1)$ -tuples by $\mathcal{C}_i(F, D, \alpha)$. So

$$\dim \mathcal{E}_{F,D,\alpha}^{(i)} = \#(\mathcal{C}_i(F, D, \alpha)).$$

Now for each $(F_1, \dots, F_{\ell-2}, T_i) \in \mathcal{C}_i(F, D, \alpha)$, we define a skew tableau T as follows:

- (a) T is of shape $F - D$.
- (b) We regard $F - D$ as a union of the skew diagrams $F_j - F_{j-1}$, $1 \leq j \leq \ell - 1$.
- (c) For each $1 \leq j \leq i - 1$, we fill the boxes in $F_j - F_{j-1}$ with the number j .
- (d) The LR tableau T_i has shape $F_i - F_{i-1}$. For $s = 1, 2$, if a box in T_i is filled with s , we replace it by $i - 1 + s$.
- (e) For each $i + 1 \leq j \leq \ell - 1$, we fill the boxes in $F_j - F_{j-1}$ with the number $j + 1$.

Then $T \in \text{ST}^{(i)}(F, D, \alpha)$. This sets up a map

$$(F_1, \dots, F_{\ell-2}, T_i) \rightarrow T$$

from $\mathcal{C}_i(F, D, \alpha)$ to $\text{ST}^{(i)}(F, D, \alpha)$. It is easy to check that this map is bijective. Thus the lemma follows. \square

Proposition 7.8 ([HL]). *For $T \in \text{ST}^{(i)}(F, D, \alpha)$, there is a polynomial $\tilde{\gamma}_T$ in $\mathcal{E}_{F,D,\alpha}^{(i)}$ such that*

$$\text{in}_\tau(\tilde{\gamma}_T) = m_{(D,F),T}.$$

In fact, for any decomposition $\Gamma = (k_1, \dots, k_c)$ of m , the paper [HL] constructed a set of highest weight vectors in $\mathcal{A}(\Gamma)$, whose leading monomials are of the form $m_{(D,F),T}$ for appropriate D, F , and T . Although the general construction of [HL] is somewhat involved, the case of $\Gamma = \Xi_{k,i}$ is relatively easy, and will be described in §7.5.

Corollary 7.9. *The set $\mathcal{B}_{F,D,\alpha}^{(i)} = \{\tilde{\gamma}_T : T \in \text{ST}^{(i)}(F, D, \alpha)\}$ is a basis for $\mathcal{E}_{F,D,\alpha}^{(i)}$.*

Proof. By Proposition 7.8, the polynomials in $\mathcal{B}_{F,D,\alpha}^{(i)}$ have distinct leading terms with respect to τ . So $\mathcal{B}_{F,D,\alpha}^{(i)}$ is linearly independent. The corollary now follows from this and Lemma 7.7. \square

Corollary 7.10. *If f is a non-zero element of $\mathcal{E}_{F,D,\alpha}^{(i)}$, then*

$$\text{in}_\tau(f) = m_{(D,F),T}$$

for some $T \in \text{ST}^{(i)}(F, D, \alpha)$.

Proof. The proof is similar to the proof of Corollary 7.6. \square

7.4. The algebra $\mathcal{A}(k, \ell)^{U_n}$. As explained in §7.1, the LR Rule is encoded in the structure of the algebra $\mathcal{A}(k, \ell)^{U_n}$. We recall that $\mathcal{A}(k, \ell)^{U_n}$ is an $A_n \times A'_k \times A'_\ell$ module, and it can be decomposed as

$$\mathcal{A}(k, \ell)^{U_n} = \sum_{F,D,E} \mathcal{E}_{F,D,E},$$

where each $\mathcal{E}_{F,D,E}$ is the $\psi_n^F \times \psi_k^D \times \psi_\ell^E$ eigenspace of $A_n \times A_k \times A_\ell$. Lemma 7.2 says that the LR Rule is equivalent to the statement $\dim \mathcal{E}_{F,D,E} = c_{D,E}^F$ for all Young diagrams D, E , and F . The final step of the proof is to construct a linearly independent subset of $\mathcal{E}_{F,D,E}$ with $c_{D,E}^F$ elements and show that it spans $\mathcal{E}_{F,D,E}$. The construction of these elements was the main result of [HTW3].

Proposition 7.11 ([HTW3]). *For each LR tableau T of shape $F - D$ and content E , there is a polynomial $\Delta_{(D,F),T}$ in $\mathcal{E}_{F,D,E}$ such that*

$$\text{in}_\tau [\Delta_{(D,F),T}] = m_{(D,F),T}.$$

We will describe the construction of the polynomial $\Delta_{(D,F),T}$ in §7.6.

Proposition 7.12. *The leading monomial of each non-zero element f in the homogeneous component $\mathcal{E}_{F,D,E}$ of $\mathcal{A}(k, \ell)^{U_n}$ is of the form $m_{(D,F),T}$, where T is an LR tableau of shape $F - D$ and content E .*

Proof. Let f be a non-zero element in $\mathcal{E}_{F,D,E}$. By equation (7.16), $f \in \mathcal{A}(\Xi_{k,i})^{U_n}$ for $i = 1, 2, \dots, \ell - 1$. In fact, since f is a eigenvector for $A_n \times A'_k \times A'_\ell$ corresponding to the character $\psi^F \times \psi_k^D \times \psi_\ell^E$, it is contained in the homogeneous component $\mathcal{E}_{F,D,E}^{(i)}$ of $\mathcal{A}(\Xi_{k,i})^{U_n}$. By Corollary 7.10,

$$\text{in}_\tau(f) = m_{(D,F),T},$$

where $T \in \text{ST}^{(i)}(F, D, E)$. Since for each $1 \leq i \leq \ell - 1$, the pair $(i, i + 1)$ satisfies YWC in T , T is an LR tableau. \square

Proof of the LR Rule. Let

$$\mathcal{B}(F, D, E) = \{\Delta_{(D,F),T} : T \in \text{LR}(F, D, E)\}.$$

By Lemma 7.2, it suffices to show that $\mathcal{B}(F, D, E)$ is a basis for $\mathcal{E}_{F,D,E}$. By Proposition 7.11, the elements of $\mathcal{B}(F, D, E)$ have distinct leading monomials, so they are linearly independent. It remains to show that $\mathcal{B}(F, D, E)$ spans $\mathcal{E}_{F,D,E}$.

Let g be a non-zero element of $\mathcal{A}_{F,D,E}$. By Proposition 7.12, $\text{in}_\tau(g) = m_{(D,F),T_1}$ for some LR tableau T_1 of shape $F - D$ and content E . We consider the polynomial

$$g_2 = g - c_1 \Delta_{(D,F),T_1},$$

where c_1 is the coefficient of the monomial $m_{(D,F),T_1}$ in g . If $g_2 = 0$, then $g = c_1 \Delta_{(D,F),T_1}$ is a linear combination of elements in $\mathcal{B}(F, D, E)$. If $g_2 \neq 0$, then $\text{in}_\tau(g_2) < \text{in}_\tau(g)$, and we consider the polynomial

$$g_3 = g_2 - c_2 \Delta_{(D,F),T_2} = g - c_1 \Delta_{(D,F),T_1} - c_2 \Delta_{(D,F),T_2},$$

where $\text{in}_\tau(g_2) = m_{(D,F),T_2}$ and c_2 is the coefficient of the monomial $m_{(D,F),T_2}$ in g_2 . Continuing this way, we see that g is a linear combination of elements of $\mathcal{B}(F, D, E)$. This shows that $\mathcal{B}(F, D, E)$ spans $\mathcal{E}_{F,D,E}$. \square

7.5. The construction of highest weight vectors in $\mathcal{A}(\Xi_{k,i})^{U_n}$. This section and §7.6 should be considered as addenda to the main discussion, which was completed in §7.4. Although we can simply appeal to [HL] for the existence of the desired highest weight vectors in $\mathcal{A}(\Xi_{k,i})^{U_n}$, the general construction of [HL] is rather involved. Therefore, it seems worthwhile pointing out here that, in the cases of the special decomposition $\Xi_{k,i}$ that we use in our proof of the LR Rule, the desired highest weight vectors can be constructed without great effort.

We observe that the polynomials $\gamma_{(p,S)}$ defined in equation (7.20) can be classified into four types:

Type 1: $i, i + 1 \notin S$. In this case, $\gamma_{(p,S)}$ is a $\text{GL}_2^{(i)}$ invariant. That is, it is a $\text{GL}_2^{(i)}$ highest weight vector with weight $\psi_2^{(0,0)}$.

Type 2: $i, i + 1 \in S$. In this case, $\gamma_{(p,S)}$ is a $\text{GL}_2^{(i)}$ relative invariant, that is, it is a $\text{GL}_2^{(i)}$ highest weight vector with weight $\psi_2^{(1,1)}$.

Type 3: $i \in S$ and $i + 1 \notin S$. In this case, $\gamma_{(p,S)}$ is a $\text{GL}_2^{(i)}$ highest weight vector with weight $\psi_2^{(1,0)}$.

Type 4: $i \notin S$ and $i + 1 \in S$. In this case, $\gamma_{(p,S)}$ is not a $\text{GL}_2^{(i)}$ highest weight vector.

In fact, if $i \in S$ and $i + 1 \notin S$ and S' is the set obtained by replacing i by $i + 1$ in S , then $\gamma_{(p,S)}$ and $\gamma_{(p,S')}$ span a $\text{GL}_2^{(i)}$ module isomorphic to \mathbb{C}^2 . We form the tensor product of two such copies of \mathbb{C}^2 . Specifically, let $1 \leq p_1, p_2 \leq k$, and for $j = 1, 2$, let

$S_j \subseteq \{1, 2, \dots, \ell\}$ be such that $i \in S_j$ and $i + 1 \notin S_j$, and denote the $\text{GL}_2^{(i)}$ module spanned by $\gamma_{(p_j, S_j)}$ and $\gamma_{(p_j, S'_j)}$ by \mathbb{C}_j^2 . If $(p_1, S_1) \neq (p_2, S_2)$, then the four-tuple of products $\{\gamma_{(p_1, S_1)}\gamma_{(p_2, S_2)}, \gamma_{(p_1, S_1)}\gamma_{(p_2, S'_2)}, \gamma_{(p_1, S'_1)}\gamma_{(p_2, S_2)}, \gamma_{(p_1, S'_1)}\gamma_{(p_2, S'_2)}\}$ spans a $\text{GL}_2^{(i)}$ module isomorphic to $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2$, and the polynomial

$$(7.36) \quad \Delta((p_1, S_1), (p_2, S_2)) = \begin{vmatrix} \gamma_{(p_1, S_1)} & \gamma_{(p_1, S'_1)} \\ \gamma_{(p_2, S_2)} & \gamma_{(p_2, S'_2)} \end{vmatrix} = \gamma_{(p_1, S_1)}\gamma_{(p_2, S'_2)} - \gamma_{(p_2, S_2)}\gamma_{(p_1, S'_1)}$$

is a $\text{GL}_2^{(i)}$ relative invariant in this module.

We now let D and F be Young diagrams such that $r(D) \leq k$ and $r(F) \leq n$, and let $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ be such that $\alpha_i \geq \alpha_{i+1}$. Assume that $\text{ST}^{(i)}(F, D, \alpha) \neq \emptyset$, and let $T \in \text{ST}^{(i)}(F, D, \alpha)$. Consider the polynomial $\gamma_T = \gamma_{T_1}\gamma_{T_2} \cdots \gamma_{T_r}$ as defined in equation (7.22). We define the polynomial $\tilde{\gamma}_T \in \mathcal{A}(\Xi_{k,i})^{U^n}$ by modifying γ_T according to the following recipe:

- (1) If no γ_{T_j} ($1 \leq j \leq r$) are of Type 4, then we set $\tilde{\gamma}_T = \gamma_T$.
- (2) Suppose some γ_{T_j} are of Type 4. Assume that T_{j_1}, \dots, T_{j_u} ($j_1 > j_2 > \dots > j_u$) and $T_{j'_1}, \dots, T_{j'_v}$ ($j'_1 > j'_2 > \dots > j'_v$) are all the columns of T which are of Type 3 and of Type 4, respectively. Since T satisfies the YWC with for i and $i + 1$, we must have $u \geq v$ and $j'_a \geq j_a$ for $1 \leq a \leq v$. Let T' be the tableau obtained by removing the columns T_{j_a} and $T_{j'_a}$, $1 \leq a \leq v$. Then we set

$$\tilde{\gamma}_T = \gamma_{T'} \prod_{a=1}^v \Delta((p_{j_a}, S_{j_a}), (p_{j'_a}, \hat{S}_{j'_a})),$$

where for each $1 \leq a \leq v$, $\hat{S}_{j'_a}$ is obtained from $S_{j'_a}$ by replacing $i + 1$ by i .

Proof of Proposition 7.8. If all γ_{T_j} ($1 \leq j \leq r$) are not of Type 4, then

$$\text{in}_\tau(\tilde{\gamma}_T) = \text{in}_\tau(\gamma_T) = m_{(D,F),T}$$

by Lemma 7.5.

Next, we assume some γ_{T_j} are of Type 4. Let $1 \leq a \leq v$, and consider the columns T_{j_a} and $T_{j'_a}$. Since the pair $(i, i + 1)$ satisfies the YWC in T , we must have $j'_a < j_a$. Since T is a semistandard tableau, a moment's thought reveals that

$$\text{in}_\tau(\Delta(p_{j_a}, S_{j_a}), (p_{j'_a}, \hat{S}_{j'_a})) = \text{in}_\tau(\gamma_{(p_{j_a}, S_{j_a})})\text{in}_\tau(\gamma_{(p_{j'_a}, S_{j'_a})}).$$

It follows from this that

$$\begin{aligned} \text{in}_\tau(\tilde{\gamma}_T) &= \text{in}_\tau(\gamma_{T'}) \prod_{a=1}^v \text{in}_\tau(\Delta((p_{j_a}, S_{j_a}), (p_{j'_a}, S_{j'_a}))) \\ &= \text{in}_\tau(\gamma_{T'}) \prod_{a=1}^v \text{in}_\tau(\gamma_{(p_{j_a}, S_{j_a})})\text{in}_\tau(\gamma_{(p_{j'_a}, S_{j'_a})}) \\ &= \text{in}_\tau(\gamma_T) \\ &= m_{(D,F),T}. \end{aligned} \quad \square$$

7.6. **Construction of highest weight vectors in $\mathcal{A}(k, \ell)^{U_n}$.** In this subsection, we briefly review the construction of the polynomial $\Delta_{(D,F),T}$ given in [HTW3]. Let $D^t = (d_1, \dots, d_r)$, $E^t = (e_1, \dots, e_s)$, and $F^t = (f_1, \dots, f_t)$. Here D^t is the conjugate diagram of D ; that is, d_1, \dots, d_r are the lengths of the columns in D . The notation E^t and F^t are interpreted similarly. Let

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1s} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2s} \\ \vdots & \vdots & & \vdots \\ \beta_{t1} & \beta_{t2} & \cdots & \beta_{ts} \end{pmatrix}$$

be a $t \times s$ matrix of indeterminates. For $1 \leq a \leq n$, $1 \leq b \leq k$, and $1 \leq c \leq \ell$, let

$$X_{a,b} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1b} \\ x_{21} & x_{22} & \cdots & x_{2b} \\ \vdots & \vdots & & \vdots \\ x_{a1} & x_{a2} & \cdots & x_{ab} \end{pmatrix} \quad \text{and} \quad Y_{a,c} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1c} \\ y_{21} & y_{22} & \cdots & y_{2c} \\ \vdots & \vdots & & \vdots \\ y_{a1} & y_{a2} & \cdots & y_{ac} \end{pmatrix}$$

be the upper left $a \times b$ (respectively $a \times c$) submatrix of the matrix X (respectively Y). Finally, we let L be the $|F| \times |F|$ matrix

$$\left(\begin{array}{cccc|cccc} X_{f_1,d_1} & 0 & \cdots & 0 & \beta_{11}Y_{f_1,e_1} & \beta_{12}Y_{f_1,e_2} & \cdots & \beta_{1s}Y_{f_1,e_s} \\ 0 & X_{f_2,d_2} & \cdots & 0 & \beta_{21}Y_{f_2,e_1} & \beta_{22}Y_{f_2,e_2} & \cdots & \beta_{2s}Y_{f_2,e_s} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & X_{f_r,d_r} & \beta_{r1}Y_{f_r,e_1} & \beta_{r2}Y_{f_r,e_2} & \cdots & \beta_{rs}Y_{f_r,e_s} \\ 0 & 0 & \cdots & 0 & \beta_{r+1,1}Y_{f_{r+1},e_1} & \beta_{r+1,2}Y_{f_{r+1},e_2} & \cdots & \beta_{r+1,s}Y_{f_{r+1},e_s} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \beta_{t1}Y_{f_t,e_1} & \beta_{t2}Y_{f_t,e_2} & \cdots & \beta_{ts}Y_{f_t,e_s} \end{array} \right).$$

The determinant of L is a polynomial in $X = (x_{ij})$, $Y = (y_{ih})$ and $\beta = (\beta_{ab})$, so it can be expressed as

$$\det L = \sum_M p_M(X, Y)\beta^M,$$

where each $M = (m_{ij})$ is a $t \times s$ matrix of non-negative integers, $\beta^M = \prod_{i,j} \beta_{ij}^{m_{ij}}$, $p_M(X, Y)$ is a polynomial in the variables $X = (x_{ij})$ and $Y = (y_{ih})$. One can associate the LR tableau T with a $t \times s$ matrix $M(T)$, and the highest weight vector $\Delta_{(D,F),T}$ is defined as

$$\Delta_{(D,F),T} = p_{M(T)}(X, Y).$$

We briefly describe how the matrix $M(T) = (m_{ij})$ is defined. By filling each column of the Young diagram E from top to bottom with consecutive positive integers starting from 1, we obtain a tableau $BT(E)$ which we call the *banal tableau* of shape E . In the paper [HTW3], a “content preserving” map from T to $BT(E)$ is defined, i.e., each cell of T is mapped to a cell in $BT(E)$ with the same value. The map can be visualized as the process of successively removing the “vertical skew strips” from T and reassembling them into columns of $BT(E)$. This is the process of standard peeling described in §5. Thus T is constructed by the reverse process of standard peeling. The contents of $BT(E)$ are moved to the skew diagrams $F - D$ one column at a time, starting from the last column of $BT(E)$. The (i, j) -th entry m_{ij} of the matrix $M(T)$ is the number of entries from the j th column of E that

get put into the i th column of $F - D$. We refer readers to the paper [HTW3] for more details.

7.7. Final remarks.

- (a) The analysis of this section shows the roles of the two conditions of semi-standardness and YWC in determining the LR tableaux. Semistandard tableaux serve to label the GL_n highest weight vectors in the tensor product $\rho_n^D \otimes \left(\bigotimes_{j=1}^{\ell} S^{a_j}\right)$. In the context of (GL_n, GL_m) -duality, this space is an eigenspace for the diagonal torus in GL_m , where $m = k + \ell$, and it is also a highest weight vector for the upper left-hand block $GL_k \subset GL_m$. The YWC then serves to describe what additional conditions are needed on the tableaux to guarantee that we also have a highest weight vector for $GL_{\ell} \subset GL_m$, where GL_{ℓ} is the lower right-hand block in GL_m .
- (b) This proof of the LR Rule sheds light on the asymmetry of the tableau description of the LR coefficients. From basic principles, it is clear that the LR coefficients $c_{D,E}^F$ are symmetric in the two factors: $c_{D,E}^F = c_{E,D}^F$. However, D and E play very different roles in the description of $c_{D,E}^F$ via LR tableaux. The construction of [HTW3] makes clear the source of the asymmetry: in defining a term order for the polynomials on $M_{n,k+\ell}$, we break the symmetry between the variables, putting the variables from $M_{n,k}$ first, and the variables from $M_{n,\ell}$ after, and this choice of order is compatible with the asymmetry in the definition of LR tableaux.
- (c) The algebra $\mathcal{A}(k, 1^{\ell}) = \mathcal{P}(M_{n,m})^{U'_k}$ can be decomposed as

$$\begin{aligned} \mathcal{P}(M_{n,m})^{U'_k} &\cong \mathcal{P}(M_{n,k})^{U'_k} \otimes \overbrace{\mathcal{P}(\mathbb{C}^n) \otimes \dots \otimes \mathcal{P}(\mathbb{C}^n)}^{\ell} \\ &\cong \sum_{\substack{r(D) \leq k \\ s_1, \dots, s_{\ell} \geq 0}} \mathcal{P}_{D, s_1, \dots, s_{\ell}}, \end{aligned}$$

where $\mathcal{P}_{D, s_1, \dots, s_{\ell}} = \rho_n^D \otimes (\rho_k^D)^{U_k} \otimes \mathcal{P}^{s_1}(\mathbb{C}^n) \otimes \dots \otimes \mathcal{P}^{s_{\ell}}(\mathbb{C}^n)$. If $s_i \geq s_{i+1}$, then the tensor product $\mathcal{P}^{s_i}(\mathbb{C}^n) \otimes \mathcal{P}^{s_{i+1}}(\mathbb{C}^n)$ contains one copy of the representation $\rho_n^{(s_i, s_{i+1})}$, so that $\mathcal{P}_{D, s_1, \dots, s_i, s_{i+1}, \dots, s_{\ell}}$ contains the tensor product

$$\rho_n^D \otimes \left(\bigotimes_{j=1}^{i-1} \mathcal{P}^{s_j}(\mathbb{C}^n)\right) \otimes \rho_n^{(s_i, s_{i+1})} \otimes \left(\bigotimes_{j=i+2}^{\ell} \mathcal{P}^{s_j}(\mathbb{C}^n)\right).$$

As a subspace of $\mathcal{P}_{D, s_1, \dots, s_i, s_{i+1}, \dots, s_{\ell}}$, this tensor product is exactly the kernel of the raising operator $E_{i, i+1} = \sum_{c=1}^n y_{ci} \frac{\partial}{\partial y_{c(i+1)}}$ from $\mathcal{P}_{D, s_1, \dots, s_i, s_{i+1}, \dots, s_{\ell}}$ to $\mathcal{P}_{D, s_1, \dots, s_i-1, s_{i+1}+1, \dots, s_{\ell}}$. Equivalently, it is the space $(\mathcal{P}_{D, s_1, \dots, s_i, s_{i+1}, \dots, s_{\ell}})^{U_2^{(i)}}$ of vectors in $\mathcal{P}_{D, s_1, \dots, s_i, s_{i+1}, \dots, s_{\ell}}$ that are fixed by $U_2^{(i)}$. The raising operator $E_{i, i+1}$ is the parallel in our context of the raising operators used in the combinatorial proofs (see for example [Stm]). Similarly, it was the raising operators for a general (Kac-Moody) Lie algebra that Littelmann ([Lim]) deformed to obtain his path description of representations.

- (d) One way of thinking about the discussion of §6 and §7 is that it provides the intrinsic meaning for the LR tableaux hinted at near the end of §5.

- (e) In [Ho2], a proof of the LR Rule for the two-rowed case is given, that does not rely on counting multiplicities. If this argument can be extended to the general two-rowed situation used in §7, it would enable a proof of the LR Rule without any counting at all. We hope to return to this issue in the future.

ACKNOWLEDGMENTS

We thank In-Sok Lee for helpful conversations and suggestions, and the referee for helpful suggestions on exposition. We are also grateful to the Institute for Advanced Studies and the Department of Mathematics of Hong Kong University of Science and Technology, and the to the Max Planck Institute of Bonn, Germany, for providing a productive working atmosphere.

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