BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 2, April 2012, Pages 333–335
S 0273-0979(2012)01369-1
Article electronically published on January 18, 2012

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
GILLES PISIER

MR1736011 (2001f:43004) 43A22; 43A46, 46B70, 46L52, 47B10

Harcharras, Asma

Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets.

Studia Mathematica **137** (1999), *no.* 3, 203–260.

This paper revisits the classical $\Lambda(p)$ sets ($p > 2$) in abelian discrete groups from a new point of view inspired by the theory of operator spaces and the non-commutative Khinchin inequalities. Let S^p denote the Schatten space; let G be a discrete group; its von Neumann algebra $\mathrm{VN}(G)$ is spanned by the left regular representation $\lambda(g),\ g \in G$, in $B(l^2(G))$; $\mathrm{VN}(G)$ is equipped with the usual trace δ_e; an operator T on $L^p(\mathrm{VN}(G))$ or S^p is called completely bounded if $T \otimes I$ is bounded on $L^p(\mathrm{VN}(G), S^p)$ or $S^p(S^p)$, respectively. $\Lambda(p)$ sets are defined as subsets of G such that $L^p(\mathrm{VN}(G))$ and $L^2(\mathrm{VN}(G))$ induce equivalent norms on the closed linear span of $\lambda(g)$, $g \in \Lambda$. If G is abelian, one recovers the usual $\Lambda(p)$ sets. As usual, a $\Lambda(p)$ set is a $\Lambda(q)$ set if $2 < q < p$. A restricted class of "completely bounded $\Lambda(p)$ sets" (denoted by $\Lambda(p)_{\mathrm{cb}}$) is defined as follows: On the closed linear span of $\lambda(g)x_g$, $g \in \Lambda$, $x_g \in S^p$, the norm induced by $L^p(\mathrm{VN}(G), S^p)$ is equivalent to $\max\{\|(\sum_{g\in\Lambda} x_g^* x_g)^{1/2}\|_{S^p}, \|(\sum_{g\in\Lambda} x_g x_g^*)^{1/2}\|_{S^p}\}$. In particular, the noncommutative Khinchin inequality for $p > 2$ means that the set of generators of $Z_2^{\mathbf{N}}$ is $\Lambda(p)_{\mathrm{cb}}$.

One of the main results of the paper is the construction of $\Lambda(2k)_{\mathrm{cb}}$ sets in every discrete group G, $2k$ being an even integer. This involves combinatorial properties introduced by W. Rudin and A. Zygmund for $G = \mathbf{Z}$ and uses either probabilistic methods or new estimates for the computation of norms in $L^{2k}(\mathrm{VN}(G), S^{2k})$. In particular, a free subset of the free group $G = \mathbf{F}^\infty$ is $\Lambda(p)_{\mathrm{cb}}$ for every $p > 2$ (this was first proved by U. Haagerup and G. Pisier). Combining the above construction with a construction of Rudin, the author exhibits $\Lambda(p)_{\mathrm{cb}}$ sets in $\mathbf{Z}$ which are not $\Lambda(q)$ for $q > p$. This implies that the $\frac{2}{p}$-interpolation space between the space of multipliers of $L^\infty(\mathbf{T})$ and the space of multipliers of $L^2(\mathbf{T})$ (which are all known to be completely bounded) strictly embeds in the space of completely bounded multipliers of $L^p(\mathbf{T})$. On the other hand, the set $\Lambda = \{2^l + 2^k \colon\ k, l \in \mathbf{N}\}$ is not a $\Lambda(p)_{\mathrm{cb}}$ set, though, as is well known, it is a $\Lambda(p)$ set for every $p > 2$. This implies the existence of multipliers of $L^p(\mathbf{T})$ which are not completely bounded.

Replacing $L^p(\mathrm{VN}(\mathbf{Z}))$ by S^p in the definition of $\Lambda(p)$ and $\Lambda(p)_{\mathrm{cb}}$ subsets of $\mathbf{Z}$ gives the new notions of $\sigma(p)$ and $\sigma(p)_{\mathrm{cb}}$ subsets of $\mathbf{N} \times \mathbf{N}$; they are related to Schur multipliers of S^p and completely bounded Schur multipliers exactly as $\Lambda(p)$ and $\Lambda(p)_{\mathrm{cb}}$ sets are related to multipliers of $L^p(\mathbf{T})$. The author obtains parallel results, noting that, if $\Lambda \subset \mathbf{N}$ is $\Lambda(p)_{\mathrm{cb}}$, then $\widehat{\Lambda} = \{(i,j) \in \mathbf{N} \times \mathbf{N} |\ i + j \in \Lambda\}$ is $\sigma(p)_{\mathrm{cb}}$.

From MathSciNet, December 2011

Françoise Lust-Piquard

MR1930886 (2004k:46096) 46L07; 46L05

Pisier, Gilles; Shlyakhtenko, Dimitri

Grothendieck's theorem for operator spaces.

Inventiones Mathematicae **150** (2002), *no.* 1, 185–217.

Let A, B be C^*-algebras. The noncommutative Grothendieck theorem asserts that for any bounded bilinear map $u\colon B \times A \to \mathbb{C}$, the associated linear mapping $\tilde{u}\colon A \to B^*$ factors through Hilbert space. More precisely, there exist a Hilbert space H and bounded linear mappings $\alpha\colon A \to H$ and $\beta\colon H \to B^*$ such that $\tilde{u} = \beta \circ \alpha$, i.e. $u(b, a) = \langle \beta(b), \alpha(a) \rangle$ for any $a \in A$ and $b \in B$, with $\|\alpha\| \, \|\beta\| \leq K \|u\|$. Here K denotes an absolute constant. This paper contains several operator space versions of that theorem and various striking applications. Two notions of complete boundedness for bilinear forms are to be considered. On the one hand, we say that u is jointly completely bounded when $\tilde{u}\colon A \to B^*$ is completely bounded, and we let $\|u\|_{\mathrm{jcb}} = \|\tilde{u}\|_{\mathrm{cb}}$ in that case. On the other hand, we say that u is completely bounded if there is a constant C such that $\left|\sum u(b_k, a_k)\right| \leq C \left\|\sum b_k b_k^*\right\|^{\frac{1}{2}} \left\|\sum a_k^* a_k\right\|^{\frac{1}{2}}$ for any finite families $(a_k)_k$ in A and $(b_k)_k$ in B. Then we let $\|u\|_{\mathrm{cb}}$ be the smallest such constant C. It is well-known that u is completely bounded if and only if $\tilde{u}$ factors through a column Hilbert space. Namely, there exist a Hilbert space H and completely bounded maps $\alpha\colon A \to H^{\mathrm{c}}$ and $\beta\colon H^{\mathrm{c}} \to B^*$ such that $\tilde{u} = \beta \circ \alpha$ and $\|\alpha\|_{\mathrm{cb}} \|\beta\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}$. In particular, complete boundedness implies joint complete boundedness and $\|u\|_{\mathrm{jcb}} \leq \|u\|_{\mathrm{cb}}$. By duality, $\tilde{u}$ factors through a row Hilbert space H^{r} if and only if the transposed bilinear form ${}^{\mathrm{t}}u\colon A \times B \to \mathbb{C}$ is completely bounded. The first main result of the paper says that if A or B is exact, and if u is jointly completely bounded, then it admits a decomposition of the form $u = v + {}^{\mathrm{t}}w$, where $v\colon B \times A \to \mathbb{C}$ and $w\colon A \times B \to \mathbb{C}$ are completely bounded. As a corollary, there exist a Hilbert space H and completely bounded linear mappings $\alpha\colon A \to H^{\mathrm{r}} \oplus H^{\mathrm{c}}$ and $\beta\colon H^{\mathrm{r}} \oplus H^{\mathrm{c}} \to B^*$ such that $\tilde{u} = \beta \circ \alpha$, with $\|\alpha\|_{\mathrm{cb}} \|\beta\|_{\mathrm{cb}} \leq K \|u\|_{\mathrm{jcb}}$. This result is also proved without any assumption on A or B, but assuming that u is approximable. Similar results are also given for bilinear forms $u\colon E \times F \to \mathbb{C}$, when E and F are exact operator spaces. As an application, the following "little Grothendieck inequality" is established. For any completely bounded map $T\colon B(H) \to OH(I)$ valued in Pisier's operator Hilbert space $OH(I)$, for any $n \geq 2$, and for any $x_1, \ldots, x_n \in B(H)$, we have

$$\sum_{k=1}^{n} \|T(x_k)\|^2 \leq K \|T\|_{\mathrm{cb}}^2 (c\mathrm{Log}(n) + 1) \Big\| \sum_{k=1}^{n} x_k \otimes \overline{x_k} \Big\|_{\min},$$

where the latter norm is computed in the tensor product $B(H) \otimes_{\min} B(H)$. Here K, c are two absolute constants. (Note that Marius Junge has recently shown that the logarithmic factor is sharp.) As another application, the authors describe both bounded and completely bounded Schur multipliers from the space of compact operators to the trace class.

Christian Le Merdy

MR2352492 (2008m:46111) 46L07; 46L89

Haagerup, Uffe; Musat, Magdalena

On the best constants in noncommutative Khintchine-type inequalities.

Journal of Functional Analysis **250** (2007), *no.* 2, 588–624.

Let S_n^1 denote the space of $n \times n$ matrices x equipped with the trace norm $x_1 = \mathrm{tr}(|x|)$. Let $(g_k)_{k\geq 1}$ be a sequence of independent standard Gaussian variables on some probability space Ω. The noncommutative Khintchine inequality on trace class (due to Lust-Piquard and Pisier) asserts that there exist two constants $C_2 \geq C_1 > 0$ such that for any $n \geq 1$ and for any finite family $(x_k)_{k\geq 1}$ in S_n^1, we have

$$C_1\|(x_k)_k\|_{\mathrm{inf}} \leq \| \sum_k g_k \otimes x_k\|_{L^1(\Omega;S_n^1)} \leq C_2\|(x_k)_k\|_{\mathrm{inf}}.$$

Here, by definition,

$$\|(x_k)_k\|_{\mathrm{inf}} = \inf\big\{\big\|(\sum_k y_k^* y_k)^{\frac{1}{2}}\big\|_1 + \big\|(\sum_k z_k z_k^*)^{\frac{1}{2}}\big\|_1\big\},$$

where the infimum runs over all finite families $(y_k)_{k\geq 1}$ and $(z_k)_{k\geq 1}$ in S_n^1 such that $x_k = y_k + z_k$ for any $k \geq 1$. This paper is devoted to the value of the best possible constants in these inequalities, and to related double estimates. It is shown that the above inequalities hold with $C_1 = 1/\sqrt{2}$ and $C_2 = 1$ and that these estimates are optimal. Moreover the same holds true if the Gaussian variables are replaced by the exponentials $e^{i2^n t}$ on the unit circle. If the Gaussian variables are replaced by Rademacher variables, the corresponding inequalities hold with $C_1 = 1/\sqrt{3}$ and $C_2 = 1$. In this case, the best possible C_1 is between $1/\sqrt{2}$ and $1/\sqrt{3}$ but its exact value is unknown. The last part of the paper is devoted to Khintchine-type inequalities established by Junge for subspaces of $R \oplus C$ (the l^∞ sum of the row and column Hilbert spaces) [see M. Junge, IMRP Int. Math. Res. Pap. **2006**, Art. ID 76978, 87 pp.; MR2268491 (2009k:46118)], and to the value of the resulting constants. As an application, it is shown that Pisier's operator Hilbert space OH cb-embeds into the predual P_* of the hyperfinite III$_1$-factor P, with cb-isomorphism constant $\leq \sqrt{2}$.

From MathSciNet, December 2011

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