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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
MANFRED DENKER

MR2195221 (2006j:01006) 01A45. 01A70, 60-03

Jacob Bernoulli

The art of conjecturing.
Together with "Letter to a friend on sets in court tennis". (English)

Translated from the Latin and with introduction and notes by Edith Dudley Sylla. *Johns Hopkins University Press*, *Baltimore, MD*, 2006. xxii+430 *pp.*

In this outstanding book, Edith Dudley Sylla provides an excellent historical introduction, translation, and commentary for a seminal work in probability theory, Jacob Bernoulli's *Ars Conjectandi* (published posthumously in 1713). It is in the *Ars Conjectandi* that we find what are now called Bernoulli trials, where the Bernoulli numbers are introduced and used to sum arbitrary powers of the integers, and, above all, where Jacob Bernoulli gave the first formulation and proof of the law of large numbers.

Although Fermat and Pascal had worked out a mathematical basis for games of chance in their correspondence and Christiaan Huygens had written a book addressing similar questions, Sylla convincingly argues that Bernoulli's work is the first to merit being called a treatment of mathematical probability. The *Ars Conjectandi* has four parts. The first is a reprint of Huygens' work but with notes, explanations, and alternative solutions that are both four times longer than the original and deeper, including the use of infinite series and logarithms. The second part is the most extensive treatment to date of permutations and combinations, causing the *Ars Conjectandi* to become the most popular work on combinatorics in the eighteenth century. The third part applies the theory of permutations and combinations to games of chance, many selected because of their mathematical interest rather than because somebody actually played them. The fourth part, the most unprecedented, discusses the application of probability theory to social, economic, and political questions, and to serve that purpose, proves the law of large numbers.

Sylla does far more in her introduction and commentary than summarize and explain Bernoulli's mathematical achievements. By her careful choice of terms for translation (and the one-to-one correspondence between Bernoulli's Latin terms and the English words she chooses for them) she makes clear that Bernoulli did not think of "probability" as a measure of the frequency of events, but instead, epistemically, that is, as the degree of certainty that we have. She sorts out the history of the most important of the mathematical Bernoullis. For instance, our Jacob (Jakob, Jacobus, James, Jacques, depending on the language) was professor of mathematics at the University of Basel from 1687 to his death in 1705. Jacob's brother was Johann Bernoulli (Johannes, John, Jean—or Johann I to distinguish him from his son Johann II, also a mathematician), best known for his work in differential calculus; he was professor of mathematics at Groningen and then Jacob's successor at Basel. Johann I was also the father of Daniel (author of Bernoulli's principle in physics and the St. Petersburg paradox). Sylla shows how the Renaissance theory of business partnerships and their possible dissolution helped shape the theory of fair division of stakes in an interrupted game that formed the basis of the work of Fermat, Pascal, and Huygens. And she unveils the theological basis of Bernoulli's discussion of "moral certainty" and its relation to the proposed application of probability theory to questions like life expectancy or the strength of evidence in legal proceedings.

Bernoulli's "principal proposition"—his version of the law of large numbers—says that in the real world, when the ratio of cases is not known ahead of time, we can nevertheless find it with any degree of accuracy desired, provided that we are willing to collect sufficiently large amounts of data. He assumes we have a general situation in which an outcome may happen in r cases and fail in s cases, represented by the binomial $(r+s)=t$. He then says that "[sufficiently] many experiments can be taken that it becomes any given number of times... more likely" that the ratio r/s "is neither more than $(r+1)/t$ nor less than $(r-1)/t$". Suppose that there are nt trials. Bernoulli then expresses $(r+s)^n t$ as a series expansion. He then shows that the largest term is included between two computable bounds, bounds which can be made as small as possible if n is taken sufficiently large. For this proof, he needs four things: how to do the series expansion, how to sum the terms of such a series, to interpret the terms of the binomial expansion to correspond to various numbers of combinations, and how to do what is essentially a limit-computation: given an arbitrary tolerance, to find n sufficiently large so that the probability of the given outcome is within that tolerance of the predicted value.

Bernoulli justifies identifying terms of the binomial expansion with possible outcomes of experiments by theology: we observe the distinct cases successively over time; God sees them all at once. This is analogous to an argument he gives in Part I of the *Ars Conjectandi* that there is no difference between throwing one die n times and throwing n dice once. Ultimately, then, Bernoulli is a determinist, concluding "If the observations of all events were continued for the whole of eternity... everything in the world would be observed to happen in fixed ratios" and follow constant laws.

Of interest also in this volume is Sylla's translation and commentary on Bernoulli's "Letter to a Friend on Sets in Court Tennis" ("le jeu de paume" in French, "royal tennis" in England), whose purpose is less to analyze the particular game than to give the first calculations of probabilities in games where skill plays a role.

Perhaps the commentary could have explained more fully the contrasting philosophy of probability of Bernoulli's contemporary Abraham De Moivre, or the links between Jacob Bernoulli's pioneering work on infinite series with his work on probability. But on the whole, this book is essential for anyone wanting to know what Jacob Bernoulli accomplished in his own terms, how his work relates to that of his predecessors, and what its mathematical importance is—all in the light of the latest and best scholarship.

From MathSciNet, April 2013

Judith V. Grabiner

MR0734760 (85e:01002) 01A05. 01A45, 01A55, 01A60, 28-03, 60-03, 82-03

Anthony Lo Bello

On the origin and history of ergodic theory.

Boll. Storia Sci. Mat. **3** (1983), *no.* 1, 37–75.

This is a very good bird's-eye view of the development of ergodic theory. The author starts with Jakob Bernoulli's law of large numbers, and shows how, by introducing the transformation known as a Bernoulli shift, it can be interpreted as expressing an equality of time average and space average. He then moves on to the introduction by the founders of statistical mechanics, Gibbs and Boltzmann, of the "ergodic hypothesis", first in a form which modern topology immediately proves false, and then again as an equality of averages, finally proved in 1932 by J. von Neumann "in the mean" and then by G. D. Birkhoff "almost everywhere". In passing, he mentions Poincare's recurrence theorem and H. Weyl's equidistribution theorem. He finally describes, up to 1982, all the later work done in ergodic theory, centering around the classification problem of ergodic measure-preserving transformations: the introduction of the notions of mixing and weakly mixing transformations, and above all the concept of entropy by Kolmogorov, showing for the first time that Bernoulli shifts were not all equivalent, and culminating in 1970 with the famous Ornstein theorem proving that entropy is enough to classify Bernoulli shifts completely. At the end some problems still open at present are mentioned.

From MathSciNet, April 2013

J. Dieudonné

MR0827905 (87h:01023) 01A45. 01A75, 60-03

Pascal Dupont; Clara Silvia Roero

The treatise *De ratiociniis in ludo aleae* of Christiaan Huygens, with the *Annotationes* of Jakob Bernoulli (*Ars conjectandi*, Part I), presented in an Italian translation, with historical and critical commentary and modern solutions (Italian).

Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (5) **8** (1984), *no.* 1-2, 3–258.

This is a translation of the first chapter of Jakob Bernoulli's *Ars conjectandi* (1713), which contains Huygens' (Latin) treatise on probability with Bernoulli's annotations. The treatise, originally written in Dutch, was translated into Latin by Huygens' teacher Van Schooten and published in 1657. The Dutch version was published later. The present Italian translation was made from the text as published in *Die Werke von Jakob Bernoulli, Band 3* [Birkhäuser, Basel, 1975; MR0505125 (58 #21349)], with B. van der Waerden as editor. It makes further use of the German translation in Ostwald's Klassiker 107 (1894), the French one in Huygens' *Oeuvres XIV* (1920) and the English one by Arbuthnot (1738). Attention to the original Dutch is given through the remarks by H. Freudenthal [Historia Math. **7** (1980), no. 2, 113–117; MR0572272 (82j:01032)].

The translation is preceded by a 17-page introduction on the history of the treatise and on the philological difficulties faced, especially in connection with the expression "expectatio". Every one of Huygens' fourteen Propositions is followed by extensive historical and mathematical commentary. This makes for a publication that is far more than a mere translation, and every student of the history of probability will do well by consulting the notes provided by the translators in this, their labor of love.

From MathSciNet, April 2013

D. J. Struik

MR2729581 (2011m:60003) 60-03. 01A50

Erwin Bolthausen

Bernoullis Gesetz der großen Zahlen (German) [Bernoulli's law of large numbers].

Elem. Math. **65** (2010), *no.* 4, 134–143.1420–8962.

Jacob Bernoulli's theorem, a statement and a proof of which appeared posthumously (in 1713) as the fourth section of the *Ars conjectandi*, is today considered to be the first version of the weak law of large numbers for random variables that can take only the values 0 or 1.

The author presents clearly, in modern terminology, the major lines of Bernoulli's original proof. He uses the term "exponential estimations", which appears in the original, for what are known today as large deviations. He seeks to show how, within Bernoulli's calculations and reflections, one can find the kernels of the theory of estimation, i.e., testing theory. Finally, he suggests some connections with subsequent developments (general mass-concentration phenomena, ergodic theory, large deviations).

From MathSciNet, April 2013

Pierre Crépel

MR1666260 (99m:60003) 60-03. 01A05

Gnedenko, B. V.

Development of probability theory. (Russian)

Outlines of the history of mathematics, Izdat. Moskov. Univ., Moscow (1997), 247–338.

One of the latest works by the prominent Russian mathematican B. V. Gnedenko (1912–1996) is devoted to the history of probability theory. In his own words, "The history of probability theory is rich in content and instructive. It shows illustratively how its essential concepts were born and methods were developed originating in the problems which the progress of humanity often faced. And yet it requires a great deal of research to be done to recover in detail its evolution and render its creators their due. Then we shall see how mankind passed from initial guesses to the knowledge more complete and perfect, how the foundation of probability theory allowed us to come from strict deterministic notions to more general stochastic concepts thus revealing new possibilities to derive deeper conclusions about the true nature of things."

The author gives a broad overview of the problems that stimulated the development of probability theory. Many crucial results and theorems that marked the evolution of this branch of mathematics are considered, taking into account their historical significance. Many forgotten names are recovered and for some very well-known results their true creators are named to restore historical justice. The author pays considerable attention to the contribution of Russian scientists to the development of probability theory.

A brief sketch of the evolution of probability theory is given in the introduction. Chapter 1 is devoted to its "prehistoric" period beginning with the problems of calculations of the number of all possible results in games of chance. It dates back to the 10th century and is closely connected to the research of G. Cardano, N. Tartaglia, G. Galilei, B. Pascal and P. Fermat.

The formation of the basic concepts of classical probability theory is considered in Chapter 2. The 18th century yields the notion of classical probability developed by J. Bernoulli and especially by A. de Moivre. Geometric probabilities originated in the works of G. de Buffon in 1733. The main properties of classical probability were formulated by T. Bayes and P. de Laplace; the notion of independence was introduced by de Moivre in 1718. §5 of Chapter 2 describes the development of limit theorems (the law of large numbers by J. Bernoulli, the de Moivre-Laplace local limit theorem (1733)). The process of axiomatization of probability theory is considered in §7 of Chapter 2. The axiomatic approaches of S. N. Bernstein, R. Mises and A. N. Kolmogorov are given and the attempts at generalizing the concepts of probability are mentioned.

Chapter 3 deals with the development of the ideas of random variable. Preliminary steps in this direction were made by de Moivre, Laplace and Gauss and the notion itself was introduced by S. Poisson (1832). The history of the laws of large numbers associated with the names of P. L. Chebyshev, S. N. Bernstein, A. Y. Khinchin and A. N. Kolmogorov is outlined in §3. The progress in the field of central limit theorems is described in §4. The works of Poisson, P. Lévy, Khinchin, Kolmogorov and Gnedenko introduced and completely classified all possible limiting distributions in the laws of large numbers (stable and infinite divisible laws). §7 treats the development of the concepts of the expectation and variance.

The last chapter of the paper is devoted to the history of stochastic processes. The first steps in this direction date back to the works of A. K. Erlang (telecommunication networks) and physicists M. Smoluchowski, M. Planck, and A. Einstein (Brownian motion). Further developments are due to N. Wiener (Gaussian processes), Kolmogorov (Markov processes), Khinchin (stationary processes), etc.

The outstanding mathematical culture of Gnedenko allows the reader to penetrate deep into the philosophical nature of the problems and tendencies of the development of probability theory.

From MathSciNet, April 2013

Werner Linde

MR2850003 (2012h:60001) 60-03. 01A50, 01A70

David R. Bellhouse

Abraham De Moivre. Setting the stage for classical probability and its applications.

CRC Press, *Boca Raton, FL*, 2011. 266 *pp.*

The book under review is the highly successful result of twenty-five years of research on the history and development of probability in Britain, which was, as the author states in the preface of the book, his first intention to write. His decision to change his approach and focus on Abraham De Moivre instead was a good one. The word “stage” in the subtitle of the book suggests to me an association to a real stage and in that sense one can clearly see throughout the book De Moivre’s skillfully conducted career as the leading actor mathematician-probabilist of eighteenth-century Britain; others around him look like supporting actors, even extras.

There are many books on the general history of probability theory [e.g., I. Todhunter, *A history of the mathematical theory of probability from the time of Pascal to that of Laplace*, Macmillan, Cambridge, 1865; reprint, Chelsea, New York, 1965; F. N. David, *Games, gods and gambling*, reprint of the 1962 original, Dover, Mineola, NY, 1998; MR1638639; A. H. Hald, *A history of probability and statistics and their applications before 1750*, Wiley Ser. Probab. Math. Statist. Probab. Math. Statist., Wiley, New York, 1990; MR1029276 (91c:01003); S. M. Stigler, *The history of statistics*, reprint of the 1986 original, Belknap Press/Harvard Univ. Press, Cambridge, MA, 1990; MR1057346 (91h:01005); L. J. Daston, *Classical probability in the Enlightenment*, Princeton Univ. Press, Princeton, NJ, 1988; MR0988886 (90g:01002)] and some papers on certain individual results of De Moivre, but only one treatise on De Moivre as a mathematician has been published [I. Schneider, Arch. History Exact Sci. **5** (1968), no. 3-4, 177–317; MR1554118]. Bellhouse's book is the first extensive intellectual biography of De Moivre; and it is excellent. Written on two levels—historical and mathematical—the book deals not only with De Moivre's work on probability, as one might expect based on the subtitle, but with his integral mathematical work as well.

The story of De Moivre's life and work is placed in the historical context of the transition from the seventeenth to the eighteenth century, with people and their customs, habits, virtues and weaknesses as the background. The time of that transition was a religiously and politically uncertain one, when both personal and intellectual connections were most relevant for success in (high, intellectual, mathematical) society. It was a time when the greatest mathematicians tried to solve publicly posed mathematical challenges and problems, when debates and even disputes about the priority of some mathematical discoveries took place, and when new mathematical fields were created and developed. All of that is brilliantly depicted throughout this book, especially in the first four chapters. Chapter 6 is devoted to De Moivre's involvement in the Leibniz-Newton priority dispute. But the major theme of the book still remains the life of De Moivre and his work in probability theory. The analysis of his mathematical achievements and results in probability is not as detailed as it is in Schneider' treatise, though. A probabilistically inclined reader may find chapters 5, 8, 11 and 12 very interesting from a mathematical point of view. Chapter 7 deals with De Moivre's mathematical research outside probability.

The author does not make explicit evaluations of De Moivre's mathematical results, his successes and the way those successes were achieved, or his relationships with his friends (I. Newton, E. Halley, W. Jones, F. Robartes, for example) and his rivals or competitors (G. Cheyne, P. R. Montmort, N. Bernoulli, and J. Bernoulli, for example). He states the facts and leaves it up to the reader to form opinions and make judgments. Everything in this book is written with mathematical clarity and exactness. Claims are proved and supported by many references to original documents, manuscript sources and comprehensive literature on De Moivre. The author does not put forward any hypotheses or speculations, and nothing is left to chance or arbitrariness. Sometimes it even seems that in order to maximize the book's objectivity, accuracy and historical definitiveness and specificity, too much surrounding data are given that may distract a less attentive reader and impede his following of the main ideas. The chapter about the two Thomases (Chapter 13), which looks like a part of the author's initial intention to write a general history of probability theory in Britain, gives such an impression. In this reviewer's opinion, it would have been better if that chapter had been omitted and had instead been published as an article in an appropriate journal. That minor quibble aside, the book is whole-heartedly recommended to all who like probability, mathematics, history, and fair speeches. This is the kind of definitive text that is a must for every history of mathematics shelf.

From MathSciNet, April 2013

Aleksandar M. Nikolić